Examine the Relationship Between Modular Form Fourier Coefficients and Limited Partition Function Congruences, Emphasizing Any Novel Number-Theoretic Implications

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Abstract

The purpose of this study is to analyze the occurrence of Congruences and to carefully evaluate different classes of limited partition functions, especially with regard to the modular behavior of these functions. The restricted partition functions of the type $p_k(q^2m + b)$ for specific primes q with k=3,5,7 are studied in this research. We examine the primes q that fulfill $p_k(q^2m + b) \equiv 0 \bmod k$ for all natural integers m for every k. The study of congruence behavior in limited partition functions is a particularly fruitful field of investigation within the realm of partition theory. This refers to the process of discovering the circumstances under which the values of these functions display predictable patterns modulo integers. We then use a standard proving technique for every case to demonstrate our findings.

Keywords: partition functions, Congruences, partition theory, restricted partition

1. Introduction:

Determining basic relationships that capture complex integer features is the core of number theory. The underlying structure of the integers can be shown by using one of the most basic mathematical concepts—writing a number as the sum of other numbers—almost surprisingly. This is called partition theory. A partition is a representation of a nonnegative number n as a sum of positive integers, known as summands, whose order is unimportant. Merca (2022) concentrated on partitions into discrete sections and Ramanujan-type congruence modulo 4. He discovered infinite families of congruence that extended traditional Ramanujan findings to new moduli and partition types by carefully examining q-series and partition identities. Smoot (2022) made two closely linked papers that greatly advanced our knowledge of infinite families of partition congruence. He investigated how modular functions may be used to construct infinite families of congruence in a systematic way in his Xiv publication and the accompanying RISC Report (2022). Smoot developed a technique called "divisibility arising from addition," which used the modular form's structure to find consistent partition divisibility patterns. His method made it possible to formulate large classes of congruence, especially those that are difficult to get using conventional combinatorial techniques. Hanson (2023) investigated the mathematical characteristics of modular form coefficients, with special attention to how they relate to partition theory. In order to derive novel congruence and divisibility conclusions, he examined the link between the coefficients of certain modular forms and partition functions. Yildiz et al. (2023) used a modular structure in the surprising yet creative fields of textile pattern creation and geometry. Their use of modular congruence principles demonstrated the wider relevance of modularity outside of conventional algebraic contexts, even though it was not directly related to number theory. Banerjee and Smoot (2023) investigated the congruence features of a particular class of partitions, known as 2-elongated planar partitions, modulo powers of 7. This was complemented by Banerjee and Smoot (2023), who applied the localization method to a genus 1 congruence family, analyzing 2-elongated plane partitions and their congruence properties with respect to powers of 7, contributing novel insights into the arithmetic of specialized partition families. Chern and Sellers (2023) investigated internal congruence's within certain constrained partition families, specifically partitions into odd parts with specified summands. They used generating function manipulations and recurrence structures to offer evidence for their identification of infinite families of congruence's modulo powers of two. Their research helped develop a more

sophisticated understanding of how congruence behavior is influenced by internal limitations inside partitions. Smoot (2024) broadened his previous research by providing a thorough framework for modular congruence family categorization. Congruencies resulting from addition were highlighted in his categorization, along with their structural reliance on modular form attributes like weight and cusp behavior. The purpose of this study is to analyze the occurrence of Congruences and to carefully evaluate different classes of limited partition functions, especially with regard to the modular behavior of these functions. In doing so, it makes a contribution to a more comprehensive understanding of the ways in which arithmetic, algebraic, and analytic features converge within the theory of modular forms and partitions.

2. Objectives of the study

- To examine the relationship between modular form Fourier coefficients and limited partition function Congruences, emphasizing any novel number-theoretic implications.
- To investigate how cusp forms, Eisenstein series, and eta-products affect limited partition generating function congruence.

3. Methodology

Theoretical Framework and Mathematical Foundations: A detailed survey of the literature on integer Congruences, modular forms, and partition theory will be the first step in the research. To provide a strong mathematical foundation, important works by Ramanujan, Atkin, Serre, and Ono as well as more recent advancements in the arithmetic of modular forms will be rigorously analyzed. We shall formalize definitions of limited partition functions, including those with odd parts, separate portions, or congruence criteria.

Construction and Analysis of Generating Functions: Building generating functions for the several limited partition functions under examination will be the next stage of the work. For example, infinite q-series or eta-products will be used to represent generating functions for partitions into odd or different sections. Where appropriate, these generating functions will subsequently be recast using modular or quasi-modular forms. To ascertain if these functions display complete or partial modularity, an analysis will be conducted of their modular features, including level, weight, and transformation behavior.

4. Data Analysis and results

We shall examine congruences of $p_3(n)\text{mod}3$ in this work. The following derivation involving the generating function of $p_3(n)$ serves as motivation for this, recalling the equation and letting k=3 shows

$$P_3(q) = \sum_{n=0}^{\infty} p_3(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{3n})}{(1-q^n)} \ldots..(1)$$

We be acquainted with that $1 - q^{3n} \equiv (1 - q^n)^3 \text{mod}3$, consequently

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^n)} \text{ mod}3 \ldots.(2)$$

This be capable of be condensed to

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv (\prod_{n=1}^{\infty} 1 - q^n)^2 \text{ mod}3 \ldots.(3)$$

Using Euler's Pentagonal Number Theorem, the equation above is transformed into

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}\right)^2 \text{ mod}3 \ldots.(4)$$

This can now be expressed as a double sum

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \sum_{k,\ell=-\infty}^{\infty} (-1)^{k+\ell} q^{\frac{k(3k-1)}{2}+\frac{\ell(3\ell-1)}{2}} \text{mod}3 \ldots.(5)$$

at what time

$$n = \frac{k(3k-1)}{2} + \frac{\ell(3\ell-1)}{2},$$

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \sum_{n=0}^{\infty} q^n \sum_{k,\ell} (-1)^{k+\ell} \text{ mod}3 \ldots..(6)$$

We can now compare coefficients to determine

$$p_3(n) \equiv \sum_{\frac{k(3k-1)}{2}+\frac{\ell(3\ell-1)}{2}=n} (-1)^{k+\ell} \text{mod}3 \ldots.(7)$$

This indicates that determining $p_3(n)$ while working modulo 3 turns into a challenge involving how to express an integer as the sum of two pentagonal numbers. In order to analyze this, we begin by writing

$n = \frac{k(3k-1)}{2} + \frac{\ell(3\ell-1)}{2}$…(8)

We then partial the four-sided figure.

$24n + 2 = (6k-1)^2 + (6\ell-1)^2$ ....(9)

The manipulation demonstrates how n can be viewed as a sum of squares problem when written as the sum of two pentagonals. Since writing an integer as a sum of squares is a well-researched topic, this is beneficial. To keep the rest of the paper simple, we'll let

$\kappa = (6k-1)$ and $\lambda = (6\ell-1)$.

$$24n + 2 = (6k-1)^2 + (6\ell-1)^2 = \kappa^2 + \lambda^2 \ldots.(10)$$

Let n be of the form $n = q^2m + b$, where q is a prime number. Every prime, q, is equal to one or three modulo four if the number 2 is taken out of the equation. Let's start by looking at primes q≡3mod4. A well-known theory about when an integer can be expressed as a sum of two squares [HW] must be introduced before we can continue.

Theorem 1: If and only if every prime factor q≡3 mod 4 occurs with an even exponent when n is stated as a product of prime-powers, then n can be written as the sum of two squares. The following theorem follows easily from this one and certain numerical data.

Theorem 2: $p_3(q^2m + b) \equiv 0\text{mod}3$, for each and every one natural statistics m, somewhere q is a prime greater than 3 on behalf of which $q \equiv 3\text{mod}4$ with b gratifying $q \| 24b + 2$.

Proof: We recognize that $n = q^2m + b$, so recall equation 10.

$24n + 2 = 24q^2m + 24b + 2 = \kappa^2 + \lambda^2$ ....(11)

In view of the fact that $q \| 24b + 2$, we be acquainted with that we can thing out accurately solitary q to acquire

$$q\left(24qm + \frac{24b+2}{q}\right) = \kappa^2 + \lambda^2 = (6k-1)^2 + (6\ell-1)^2 \ldots.(12)$$

But for the reason that $\text{ord}_q(n) = 1$ we can be relevant Theorem 1. This let know us that there survive no (κ, λ), as well as for this reason no (k, ℓ), that persuade equation (12).

$p_3(n) \equiv \sum_{\frac{k(3k-1)}{2} + \frac{\ell(3\ell-1)}{2} = n} (-1)^{k+\ell} \text{mod}3$….(13)

Since the sum is exactly zero and contains no terms,

$p_3(q^2m + b) \equiv 0 \text{ mod}3$….(14)

We must now investigate primes congruent to one modulo four after characterizing the congruences for primes congruent to three modulo four. The fact that we cannot just apply Theorem 1 will make this considerably more difficult. It turns out that these primes must be taken into account modulo 12. We know that q≡1,5mod12 for any prime q≡1mod4. We may easily observe from numerical data that no congruence features are evident when q≡1mod12. Thus, we wish to examine the situation in which q≡5mod12. After proving a particular congruence, we will apply our methodology to all primes q≡5mod12.

Theorem 3: $p_3(5^2m + b) \equiv 0\text{mod}3$, in support of all ordinary numbers m, with b enjoyable $5 \| 24b + 2$.

Proof: We place $of\ protection = 5^2m + b$. we perceive that

$24n + 2 = 600m + 24b + 2 = 5\left(120m + \frac{24b+2}{5}\right) = \kappa^2 + \lambda^2$,….(15)

everyplace $\frac{24b+2}{5} \in \mathbb{Z}$ as $5 \| 24b + 2$.

This acquaint with us that 5 divides $\kappa^2 + \lambda^2$. Regrettably significant that a numeral separate a sum be not awfully ready to lend a hand, so we wish for to inscribe $\kappa^2 + \lambda^2$ as a creation. In support of this we believe the Gaussian Integers in addition to write $\kappa^2 + \lambda^2 = (\kappa + \iota\lambda)(\kappa -$

$\iota\lambda)$. So $5 \mid (\kappa+\iota\lambda)(\kappa-\iota\lambda)$. We know on the other hand that $5 \nmid (\kappa+\iota\lambda)$ in addition to $5 \nmid (\kappa-\iota\lambda)$ since if five separate one it divide the extra since they are multifaceted conjugates. This forces $5^2 \mid (\kappa+\iota\lambda)(\kappa-\iota\lambda)$, which contradict our form that $5\|24b+2$. It is broadly identified that five can be printed as a creation of two primes in the Gaussians, namely $5=(2+\iota)(2-\iota)$.

Thus we distinguish that $(2+\iota)(2-\iota) \mid (\kappa+\iota\lambda)(\kappa-\iota\lambda)$.

Now assume that $(2+\iota) \mid (\kappa+\iota\lambda)$.

So

$(\kappa+\iota\lambda)=(2+\iota)(\kappa'+\iota\lambda')$….(16)

We can come across a conjugate key when $(2-\iota) \mid (\kappa^*+\iota\lambda^*)=(2-\iota)(\kappa'+\iota\lambda')$. To perceive how this relate to our innovative explanation we grow by $\frac{2-\iota}{2+\iota}$. We know that $\frac{2-\iota}{2+\iota}=\frac{3-4\iota}{5}$. The real denominator willpower not involve the conjugation so we find:

$$\begin{aligned}\kappa^*+\iota\lambda^* &= [(6k-1)+\iota(6\ell-1)][3-4\iota]\\ &= [18k+24\ell-7]+\iota[18\ell-24k+1] \quad \text{...(17)}\\ &= [6(3k+4\ell-1)-1]+\iota[6(-3\ell+4k)-1].\end{aligned}$$

Consequently

$k^*=3k+4\ell-1$ and $\ell^*=-3\ell+4k$….(18)

For that reason

$(-1)^{k^*+\ell^*}=(-1)^{7k+\ell-1}=(-1)^{k+\ell-1}=-\left[(-1)^{k+\ell}\right]$…(19)

In view of the fact that

$p_3(n) \equiv \sum_{\frac{k(3k-1)}{2}+\frac{\ell(3\ell-1)}{2}=n} (-1)^{k+\ell} \bmod 3$….(20)

$p_3(5^2m+b) \equiv 0 \bmod 3$ .....(21)

It now appears possible to generalize this method to other primes, $q \equiv 5 \bmod 12$, besides 5 . This can be done, but first we need to establish a Lemma.

Theorem 4: $p_3(q^2m+b) \equiv 0 \bmod 3$, in support of each and every one accepted numbers m, where q is a prime pro which $q \equiv 5 \bmod 12$ by b rewarding $q\|24b+2$.

Proof: Let $n=q^2m+b$ where q is a prime in support of which $q \equiv 5 \bmod 12$.

Give permission

$q\|24b+2$….(21)

Subsequently

$$\begin{aligned}24n+2 &= 24q^2m+24b+2,\\ &= q\left(24qm+\frac{24b+2}{q}\right). \quad \text{...(22)}\end{aligned}$$

We can feature out this q as we be acquainted with that $\frac{24b+2}{q}$ is an numeral since $q\|24b+2$.

Consequently

$q\left(24qm+\frac{24b+2}{q}\right)=(\kappa+\iota\lambda)(\kappa-\iota\lambda)$ .....(23)

Subsequently

$q \mid (\kappa+\iota\lambda)(\kappa-\iota\lambda)$….(24)

Other than q cannot separate entirely into simply one of them because they are compound conjugates, condition $q \mid (\kappa+\iota\lambda)$ afterward $q \mid (\kappa-\iota\lambda)$, but this imply that we know how to thing out a q^2 on the left side. This contradict our state that $q\|24b+2$. consequently q must thing, but given that it is prime we call for to make use of the Gaussian integers $\mathbb{Z}[\iota]$. because q is prime and $q \equiv 1 \bmod 4$, we be on familiar terms with that present exist integers x, y s.t.

$q=(x+\iota y)(x-\iota y)=x^2+y^2$….(25)

Subsequently

$(x+\iota y)(x-\iota y) \mid (\kappa+\iota\lambda)(\kappa-\iota\lambda)$ ...(26)

At this summit we be familiar with moreover $(x+\iota y) \mid (\kappa+\iota\lambda)$ which imply $(x-\iota y) \mid (\kappa-$

$\iota\lambda)$, or $(x+\iota y) \mid (\kappa-\iota\lambda)$ which involve $(x-\iota y) \mid (\kappa+\iota\lambda)$. The selection is chance. understand that $(x+\iota y) \mid (\kappa+\iota\lambda)$. Then $\kappa+\iota\lambda=(x+\iota y)(\kappa'+\iota\lambda')$. given that this preference was uninformed we identify there exist a conjugate explanation (κ^*,λ^*) in support of which $(x-\iota y) \mid (\kappa^*+\iota\lambda^*)=(x-\iota y)(\kappa'+\iota\lambda')$. consequently we can grow $\kappa+\iota\lambda=(x+\iota y)(\kappa'+\iota\lambda')$ by $\frac{x-\iota y}{x+\iota y}$ to distinguish how the conjugate explanation relate to the novel.

$(\kappa^*+\iota\lambda^*)=(\kappa+\iota\lambda)\left(\frac{x-\iota y}{x+\iota y}\right)$...(27)

In addition, we are familiar with that

$\frac{x-\iota y}{x+\iota y}=\frac{x^2-y^2-2xy\iota}{x^2+y^2}$.....(28)

Since the denominator is genuine, the conjugation won't be affected. For simplicity let $a=x^2-y^2$ and $b=2xy$....(29)

Accordingly

$$(\kappa^*+\iota\lambda^*)=(\kappa+\iota\lambda)(a-b\iota)$$
$$=[(6k-1)+\iota(6\ell-1)][a-b\iota] \quad ...(30)$$
$$=[6(ka+\ell b)-a-b]+\iota[6(\ell a-kb)-a+b].$$

We be acquainted with that $-a\pm b\equiv\pm 1 \bmod 6$. exclusive of losing sweeping statement, we can assume $-a-b\equiv -1 \bmod 6$. This as well imply that $a\equiv 3 \bmod 6, b\equiv 4 \bmod 6$ in addition to $-a+b\equiv 1 \bmod 6$.

Consequently $-a-b+1\equiv 0 \bmod 6$ and $-a+b-1\equiv 0 \bmod 6$.

We can issue out a 6 and rephrase our equation as

$\kappa^*+\iota\lambda^*=\left[6\left(ka+\ell b+\frac{-a-b+1}{6}\right)-1\right]+\iota\left[6\left(-\ell a+kb+\frac{a-b+1}{6}\right)-1\right]$....(31)

at this instant we encompass our equation in the accurate form to comprehend that since $\kappa^*=(6k^*-1)$ and $\lambda^*=(6\ell^*-1)$, we be acquainted with that

$$k^*=ka+\ell b+\frac{-a-b+1}{6}$$
$$\ell^*=-\ell a+kb+\frac{a-b+1}{6}. \quad ...(32)$$

It become clear that we desire to compare the equality of $k+\ell$ by means of $k^*+\ell^*$.

$$k^*+\ell^*=(a+b)k+(b-a)\ell+\frac{1-b}{3}$$

First we could do with to make sure that $\frac{1-b}{3}$ is an integer. former we well-known that as $-a-b\equiv -1 \bmod 6$ then $b\equiv 4 \bmod 6$, consequently $1-b\equiv -3\equiv 3 \bmod 6$. Consequently $\frac{1-b}{3}$ is an integer. Now we be obliged to consider $k^*+\ell^* \bmod 2$. We bring to mind that $b=2xy$, consequently b is even. In addition recall $a=x^2-y^2$. Now because q is odd we be familiar with $q=x^2+y^2\equiv 1 \bmod 2$, as a result $a=x^2-y^2\equiv(x^2+y^2)-2y^2\equiv 1-0\equiv 1 \bmod 2$, so a is odd. Therefore $(a+b)\equiv(b-a)\equiv 1 \bmod 2$. We also know $\frac{1-b}{3}\equiv(1-b) \bmod 2$. But since $(1-b)\equiv 3 \bmod 6$ we know $(1-b)$ is odd, so $(1-b)\equiv 1 \bmod 2$. So reducing $k^*+\ell^* \bmod 2$ we see that $k+\ell\equiv k^*+\ell^*+1 \bmod 2$.

for that reason

$(-1)^{k+\ell}=-\left[(-1)^{k^*+\ell^*}\right]$ ....(33)

For each explanation (k,ℓ) there unavoidably exists a conjugate solution (k^*,ℓ^*). come together this with the consequence higher than we see that

$$p_3(q^2m+b)=\sum_{\frac{k(3k-1)}{2}+\frac{\ell(3\ell-1)}{2}=n}(-1)^{k+\ell}\equiv 0 \bmod 3$$

Our two primary theorems can be combined to produce a general result for the requirements on q that permit it to satisfy congruence relations.

5. Conclusion

Congruences for restricted partition functions reveal divisibility properties and patterns in partitions with constraints, often proven using Ramanujan's theta-function identities or connections to other combinatorial identities like Euler's Pentagonal Number Theorem. Researchers study these congruences to find infinite families of divisibility results for various restricted functions, similar to the classic Ramanujan congruences for the unrestricted partition function p(n), and these findings contribute to a deeper understanding of the arithmetic properties of partitions.

6. Limitations

Congruences for restricted partition functions reveal patterns in these functions' behavior modulo certain numbers, similar to Ramanujan's famous congruences for the unrestricted partition function p(n). The term "limitation" relates to the challenges and complexities in finding such congruences for restricted cases, particularly the absence of general Ramanujan-type congruences for many restricted functions and the intricate nature of the restrictions themselves, which often leads to specialized, case-by-case analysis.

7. REFERENCES

- Banerjee, K., and Smoot, N. A. (2023). 2-Elongated Plane Partitions and Powers of 7: The Localization Method Applied to a Genus 1 Congruence Family. *RISC Report Series*, 23-10.
- Banerjee, K., and Smoot, N. A. (2023). 2-elongated plane partitions and powers of 7: the localization method applied to a genus 1 congruence family. *arXiv preprint arXiv:2306.15594.*, 1-12.
- Chern, S., and Sellers, J. A. (2023). An infinite family of internal congruences modulo powers of 2 for partitions into odd parts with designated summands. *arXiv preprint arXiv:2308.04348*, 1-25.
- Merca, M. (2022). Ramanujan-type congruences modulo 4 for partitions into distinct parts. *An. St. Univ. Ovidius Constanta*, 30(3), 185-199.
- Meruliya, V., Mukhi, S., and Singh, P. (2021). Poincaré series, 3d gravity and averages of rational CFT. *Journal of High Energy Physics*, 4, 1-49.
- Smoot, N. A. (2022). Divisibility Arising From Addition: The Application of Modular Functions to Infinite Partition Congruence Families. *arXiv preprint arXiv:2209.05896*, 27-36.
- Smoot, N. A. (2022). Divisibility arising from addition: the application of modular functions to infinite partition congruence families. *RISC Report Series*, 22-18.
- Smoot, N. A. (2024). On the Classification of Modular Congruence Families. *arXiv preprint arXiv:2403.10681*, 147-152.
- Yildiz, T., Akleman, E., Krishnamurthy, V., and Ebert, M. (2023). A modular approach for creation of any bi-axial woven structure with congruent tiles. *Computers & Graphics*, 114, 357-367.