Partition Theory's Modularity Framework by Exploring Modular Curves and Galois Representations for Extensions of Known Congruences

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Abstract

The question of how these new families might be explained combinatorially arose once more. One of this thesis's key theorems establishes that the same crank defined by Andrews-Garvan is a sort of "universal" statistic for partition congruences since it satisfies exactly the same kind of congruences as those discovered by Ono for the partition function. The main result here is the existence of positive density sets of primes ℓ for which almost all of the coefficients of these quotients vanish modulo powers of ℓ. The methods employed here readily extend to more general classes of modular forms, and these results illustrate two significant uses of modular forms in number theory. These and many other number-theoretic functions' coefficients can be effectively understood using the analytic modular transformation features.

Keywords: partition functions, Congruences, partition theory, restricted partition

1. INTRODUCTION:

Modular forms have as many and as significant applications in contemporary number theory. The study of theta functions, the generating functions for class numbers, representations by quadratic forms, the analytic features modular L-functions, and elliptic functions are some of the early advances in the field. Since the 19th century, these fields have dominated number theory study and still have an impact on ongoing studies in the field. In recent years, modular forms have remained fundamentally important in the proofs of numerous ground-breaking results, such as the proof of Fermat's Last Theorem by Wiles and the works of Fields medalists Borcherds and Deligne on the relationships between the coefficients of modular forms and Galois representations. The arithmetic of the coefficients of modular forms is crucial because in many of these contexts, modular forms emerge as the generating functions of number theoretic functions. In general, a function on the complex upper-half plane that satisfies specific analytic transformation criteria and possesses a Fourier series is called a modular form. The ability of modular transformation laws to "translate" the function's analytic features into striking combinatorial correlations among the coefficients is what gives modular forms their strength in number theory applications. This thesis's findings build on some of the author's earlier publications that emphasize the interaction between two well-known examples of modular forms' analytic and combinatorial characteristics. Saeidi et al. (2015) proposed a search-based multi-view clustering approach for analyzing and evolving large-scale software systems. This research addressed the growing need for scalable techniques in software architecture, particularly in legacy and complex systems. Ahlgren and Kim (2015) made significant contributions to number theory through their study of mock theta functions and weakly holomorphic modular forms modulo 2 and 3. Ahlgren and Kim (2015) investigated the modular behavior of mathematical objects in their study of mock theta functions and weakly holomorphic modular forms modulo 2 and 3. Wang (2017) in his doctoral dissertation, investigated the arithmetic properties of partitions and derived several Hecke-Rogers-type identities. Similarly, Kim (2010) examined the arithmetic aspects of partition functions through the lens of q-combinatory, offering new identities and congruence that highlighted the subtle connections between partition theory and modular forms. Wang (2017) examined congruence for constrained bipartitions modulo powers of 5. His findings demonstrated how important q-series factorization is for comprehending congruence behavior at a greater level of detail. Dyer et al. (2018) examined partition functions in 2D CFTs, or flavored 2D conformal field theories. Hanson (2023) investigated the mathematical characteristics of modular form coefficients, with

special attention to how they relate to partition theory. Yildiz et al. (2023) used a modular structure in the surprising yet creative fields of textile pattern creation and geometry. Their use of modular congruence principles demonstrated the wider relevance of modularity outside of conventional algebraic contexts, even though it was not directly related to number theory. Banerjee and Smoot (2023) investigated the congruence features of a particular class of partitions, known as 2-elongated planar partitions, modulo powers of 7. This was complemented by Banerjee and Smoot (2023), who applied the localization method to a genus 1 congruence family, analyzing 2-elongated plane partitions and their congruence properties with respect to powers of 7, contributing novel insights into the arithmetic of specialized partition families. Chern and Sellers (2023) investigated internal congruence's within certain constrained partition families, specifically partitions into odd parts with specified summands. Smoot (2024) broadened his previous research by providing a thorough framework for modular congruence family categorization. Congruencies resulting from addition were highlighted in his categorization, along with their structural reliance on modular form attributes like weight and cusp behavior.

2. OBJECTIVES OF THE STUDY

- To investigate how cusp forms, Eisenstein series, and eta-products affect limited partition generating function congruence.
- To expand partition theory's modularity framework by exploring modular curves and Galois representations for extensions of known Congruences.

3. NEED OF THE STUDY

In number theory and Combinatorics, partition functions are essential, and their congruence properties particularly those identified by Ramanujan have stimulated a great deal of study. Even though classical partition functions have been extensively researched, little is known about the congruence behavior of limited partition functions, such as those with odd or different components or certain modular constraints, especially when looking at them from the standpoint of modular forms. Furthermore, current approaches often lack sufficient integration between theory and computing and are either strictly theoretical or computational In order to solve this issue, the current work looks at how congruence relations are governed by modular characteristics of generating functions and if new Ramanujan-type Congruences may be constructed for larger classes of confined partitions.

4. RESEARCH METHODOLOGY

Theoretical Framework and Mathematical Foundations: A detailed survey of the literature on integer Congruences, modular forms, and partition theory will be the first step in the research. To provide a strong mathematical foundation, important works by Ramanujan, Atkin, Serre, and Ono as well as more recent advancements in the arithmetic of modular forms will be rigorously analyzed. We shall formalize definitions of limited partition functions, including those with odd parts, separate portions, or congruence criteria. To provide a solid conceptual foundation for deriving Congruences, fundamental ideas in the theory of modular forms, q-series, eta-quotients, and Hecke algebras will also be examined.

Computational Experimentation and Verification: Computational methods will be essential to the research because of the intricacy of modular forms and the behavior of their coefficients. Large sequences of the partition functions' coefficients will be calculated and analyzed using symbolic algebra systems like Sage Math, Mathematica, and PARI/GP. Potential Congruences will be found and theoretical predictions will be verified with the aid of these computer experiments. Candidate Congruences will be postulated and then formalized mathematically via the recognition of numerical patterns. To guarantee accuracy and dependability, cross-validation will be conducted using known Congruences from the literature as benchmarks.

Comparative and Generalization Studies: Congruence behavior across several families of constrained partition functions will be compared in the study's final phase. This will assist in figuring out if these Congruences are governed by any unifying modular rules. Whenever

feasible, efforts will be made to extend recently found Congruences to Moduli larger than tiny primes or to more general classes of partitions. Possible links with Galois representations, harmonic Maass forms, or fake modular forms may also be investigated. This more comprehensive modular approach will seek to improve the research's theoretical depth and long-term mathematical importance.

5. DATA ANALYSIS AND RESULTS

Quotients of Eisenstein series

Inverse power series expansions

The congruences of Theorem provide certain arithmetic progressions such that $a_i(n) \equiv 0 (\mathrm{mod} \ell^a)$, where $\ell \in \{3,7\}$ and a is a small, positive integer. Berndt and Yee's proof relies on the fact that each of the series has the form

$$F(q) = 1 + M \sum_{n=1}^{\infty} a(n) q^n \ldots . (1)$$

where M is some rational number. Setting $G(q) := 1/F(q)$ and observing that $G(q)F(q) = 1$ leads to a simple functional equation for the power series of $G(q)$, namely

$$G(q) = 1 - G(q) \left(M \sum_{n=1}^{\infty} a(n) q^n \right) \ldots . (2)$$

This can then be iterated to give an " M-adic" expansion

$$G(q) = 1 - M \sum_{n_1=1}^{\infty} a(n_1) q^{n_1} + M^2 \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} a(n_1) a(n_2) q^{n_1+n_2} + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k M^k \sum_{n_1,\ldots,n_k=1}^{\infty} a(n_1) \cdots a(n_k) q^{n_1+\cdots+n_k} \ldots (3)$$

The generating function for overpartitions is

$$\sum_{n \geq 0} \bar{p}(n) = \prod_{n \geq 1} \frac{1+q^n}{1-q^n} = \left(1 + 2 \sum_{n \geq 0} (-1)^n q^{n^2} \right)^{-1}$$

This means that there is an inverse series expansion of the form (1) with $M = 2$. The density then follows from asymptotics and parity properties for the number of representations of integers by simple diagonal quadratic forms. However, in both of the settings just described the method was clearly restricted to primes that divide the leading factor M, and a different approach is required to obtain divisibility properties for infinite classes of primes. It is here that the theory of modular forms modulo ℓ can be applied.

The existence of $F_m(z)$

Lemma: If τ is sufficiently large, then there is some $\lambda \geq 1$ such that

$$\frac{\widetilde{G_m}(24z)}{\eta^{\ell}(24\ell z)} \cdot E_{j+1}(24z)^{\ell^{\tau}} \in S_{\lambda+\frac{1}{2}}\left(\Gamma_1(576\ell^2 N^2)\right) \ldots . (4)$$

Proof: Basic facts about modular forms along with Proposition, and (3) imply that $\widetilde{G_m}(24z) \in M^!_{(\ell+1)/2}\left(\Gamma_1(48\ell^2 N^2)\right)$. Combining this with the eta-product factors makes it clear that $\Gamma_1(576\ell^2 N^2)$ is the appropriate congruence subgroup, we know that $E_{j+1}(z)$ vanishes at each cusp $\frac{a}{c}$ where $\ell N \nmid c$. Once τ is taken to be sufficiently large, it only remains to be shown that $\widetilde{G_m}(z)/\eta^{\ell}(\ell z)$ vanishes at each cusp $\frac{a}{c}$ with $\ell N \mid c$, as replacing z by $z/24$ does not affect the signs of the cusp orders. Such a cusp $\frac{a}{c}$ is associated with a matrix of the form

$$A_{a,c} := \begin{pmatrix} \bar{a} & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \Gamma_0(\ell N), \ldots (5)$$

since $e^{2\pi i (A_{a,c} z)}$ vanishes as $z \to \frac{a}{c}$. However, since the behavior at $\frac{a}{c}$ is completely determined

by the value of c for the relevant modular forms, a slight abuse of notation can be adopted whereby the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is used to compute the order at $\frac{a}{c}$ (even though this latter matrix is actually $A_{\bar{a},c}$). The expansion of the denominator function at the cusp $\frac{a}{c}$ with $\ell N \mid c$ is

$$\frac{1}{\eta^{\ell}(\ell z)}\Big|_{\ell/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = q^{\ell^2/24} + \cdots (6)$$

Thus the proof of the lemma hinges on showing that the expansion of $\widetilde{G_m}(z)$ at $\frac{a}{c}$ is $(* \, q^h + \cdots)$ for some $h > \ell^2/24$

$$G_m(z)|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{-i}{2\pi}\left(\sum_{s=1}^{N-1} \frac{\eta^{\ell}(\ell z)}{\eta(z)} \cdot \frac{\omega_s \zeta^{-ms}}{t_{0,s}(z)}\right)\Big|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \frac{-i}{2\pi}\left(\frac{d}{\ell}\right)\frac{\eta^{\ell}(\ell z)}{\eta(z)}\sum_{s=1}^{N-1} \frac{\omega_s \zeta^{-ms}}{\beta_s t_{0,\overline{ds}}(z)} \ldots . (7)$$

so that

$$t_{0,s}(z)\big|_{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta_s \cdot t_{0,\overline{ds}}(z) \ldots (8)$$

To find the expansion of $G_m(z) \otimes \left(\frac{\cdot}{\ell}\right)$, first observe that for any $v' \equiv d^2 v (\mathrm{mod} \ell)$ there is a commutation relation

$$\begin{pmatrix} 1 & -v/\ell \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \ldots . (9)$$

where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \begin{pmatrix} a - cv/\ell & b - cvv'/\ell^2 + (av' - dv)/\ell \\ c & d + cv'/\ell \end{pmatrix} \in \Gamma_0(\ell N) \ldots (10)$$

set $g := g_{\ell}$

$$\left(G_m(z) \otimes \left(\frac{\cdot}{\ell}\right)\right)\Big|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ldots (11)$$

$$= \frac{g}{\ell}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right) G_m(z)\left|\begin{pmatrix} 1 & -\frac{v}{\ell} \\ 0 & 1 \end{pmatrix}\right|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ldots . (12)$$

$$= \frac{g}{\ell}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right) G_m(z)\left|\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right|_{\frac{\ell+1}{2}}\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \ldots (13)$$

$$= \frac{-ig}{2\pi\ell}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right)\left(\sum_{s=1}^{N-1} \frac{\eta^{\ell}(\ell z)}{\eta(z)} \cdot \frac{\omega_s \zeta^{-ms}}{t_{0,s}(z)}\right)\Big|_{\frac{\ell+1}{2}}\left|\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right|\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \ldots (14)$$

Now this can be evaluated using the modular transformation properties of $t_{0,s}(z)$ and $\eta^{\ell}(\ell z)/\eta(z)$.

$$\left(G_m(z) \otimes \left(\frac{\cdot}{\ell}\right)\right)\Big|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ldots (15)$$

$$= \frac{-ig}{2\pi\ell}\left(\frac{d'}{\ell}\right)\sum_{s=1}^{N-1}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right)\frac{\eta^{\ell}(\ell z)}{\eta(z)} \cdot \frac{\omega_s \zeta^{-ms}}{\beta_s' t_{0,\overline{d's}}(z)}\left|\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}\right| \ldots (16)$$

Thus the expansion of (16) begins as

$$\frac{-i}{2\pi}\left(\frac{d}{\ell}\right)\left(\sum_{s=1}^{N-1}\frac{\omega_s\zeta^{-ms}}{\omega_{\overline{ds}}\cdot\beta_s}q^{\delta_\ell}+\cdots\right)\ldots(17)$$

and (17) begins with

$$\frac{-ig}{2\pi\ell}\left(\frac{d}{\ell}\right)\left(\sum_{s=1}^{N-1}\frac{\omega_s\zeta^{-ms}}{\omega_{\overline{ds}}\cdot\beta_s}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right)q^{\delta_\ell}\left|\begin{pmatrix}1 & -v'/\ell\\ 0 & 1\end{pmatrix}+\cdots\right.\right)\ldots(18)$$

$$=\frac{-ig}{2\pi\ell}\left(\frac{d}{\ell}\right)\left(q^{\delta_\ell}\sum_{s=1}^{N-1}\frac{\omega_s\zeta^{-ms}}{\omega_{\overline{ds}}\cdot\beta_s}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right)e^{-2\pi iv'\delta_\ell/\ell}+\cdots\right)..(19)$$

Multiplying the first term of the above equation by $-\epsilon_\ell$ gives precisely the negative of the displayed term in (17), and hence they cancel in $\widetilde{G_m}(z)$. The cusp expansion has only integral powers of q due to the series expansion of the Klein forms and eta-quotient in (18) and (19), and therefore must have the form ($*\, q^{\delta_\ell+1}+\cdots$). Since $\delta_\ell+1>\ell^2/24$, the proof is complete.

6. CONCLUSION

The divisibility properties of the coefficients of Eisenstein series and the partition function are primary examples of the importance of the arithmetic of modular forms in number theory, and the results of this thesis exhibit the ongoing development of this field. Holomorphic modular forms are very special analytic objects, and therefore it's not surprising that many of the modular forms that arise as generating functions of natural number-theoretic objects lie in spaces of modular forms with fewer analytic conditions. However, the ℓ-adic properties of these functions are often still quite nice, and because the theory of ℓ-adic modular forms also encompasses a wider class of functions, the arithmetic of their coefficients have become more accessible to study.

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