On A Subclass of a Multivalent Function with Negative Coefficients Defined By Generalized Salagean Operator

P. Dixit, Department of Mathematics, U.I.E.T. Campus, C.S.J.M. University, Kanpur-208024, (U.P.), India
R. Agarwal, Department of Computer Applications, U.I.E.T. Campus, C.S.J.M. University, Kanpur-208024, (U.P.), India
S. Porwal, Department of Mathematics, Ram Sahay Government Degree College, Bairi- Shivrajpur, Kanpur-209205, (U.P.), India
N. Kumar, Department of Mathematics, U.I.E.T. Campus, C.S.J.M. University, Kanpur-208024, (U.P.), India
E-mail: dixit_poonam14@rediffmail.com, ritesh.840@rediffmail.com, saurabhjcb@rediffmail.com

Abstract

In this paper, we introduce and study a new subclass of analytic function with negative coefficients which is defined by using generalized Salagean operator in the unit disc. By giving specific values of we obtain important classes studied by various researchers in earlier work. In fact, an attempt has been made to have a unified and detailed study of uniformly convex functions. The result presented by here included coefficient estimates. distortion theorem, closure theorem etc. of several functions belonging to this class.

Keywords: analytic, univalent, Salagean operator Uniformly convex function, starlike functions, multivalent functions.

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1 Introduction

Let $A_{j,}\ (j \in N = 1, 2, 3, \ldots\ldots\ldots\ldots)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2}^{\infty} a_{k+p-1} z^{k+p-1},$$

which are analytic in the unit disc $E = \{\ z \in C\ |z| < 1\}$. For function $f(z)$ in A_j, Al-Oboudi [1] defined

$$D_\lambda^0\ f(z) = f(z)$$

$$D_\lambda^1\ f(z) = (1-\lambda)\, f(z) + \lambda\, z\, f'(z) = D_\lambda\, f(z)\, \lambda \geq 0$$

$$D_\lambda^n\ f(z) = D_\lambda\, D_\lambda^{n-1}\ f(z)$$

Then $D_\lambda^n\ f(z) = z^p + \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}$

If we put $\lambda = 1$, we have Salagean operator introduced by Salagean [9].

With help of Salagean operator D_λ^n, we say that the function $f(z)$ belonging to A_j is in $S(n,\ m,\ j,\ \alpha,\ \lambda)$, if and only if

$$\operatorname{Re}\left[\frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)}\right] \geq \alpha \left|\frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} - 1\right|$$

for some $\alpha \geq 0$ and for all $z \in E$

Let T_j denote the subclass of consisting of functions of the form

$$f(z) = z^p - \sum_{n=2}^{\infty} a_{k+p-1} z^{k+p-1},\ (a_{k+p-1} \geq 0) \tag{1}$$

Further, we define the class $T(n,\ m,\ j,\ \alpha,\ \lambda)$ by

$T(n,\ m,\ j,\ \alpha,\ \lambda) = S(n,\ m,\ j,\ \alpha,\ \lambda) \cap T_j$.

By giving a specific values of $n,\ m,\ j,\ \alpha$ and λ, we obtain imported classes studied by various researchers in earlier works (see [2], [3], [5], [6], [8], [11]).

In view of this remark we see that a study of the class $T(n,\ m,\ j,\ \alpha,\ \lambda)$ leads to unified results on properties of various subclasses of multivalent function.

2. Main Results

In this section, we give some result for the class. Our first result is contained in the following theorem.

Theorem 2.1. Let the function $f(z)$ be defined by (1), then $f(z)$ belong to $T(n, m, j, \alpha, \lambda)$, if and only if

$$\sum_{k=2}^{\infty}[1+(k+p-2)\lambda]^{n}\left[(1+(k+p-2)\lambda)^{m}(\alpha+1)-\alpha\right]a_{k+p-1}\leq 1 \qquad (2)$$

The result is sharp.

Proof. Assume that $f(z)\in T(n, m, j, \alpha, \lambda)$, Then by definition

$$\operatorname{Re}\left[\frac{D_{\lambda}^{n+m}f(z)}{D_{\lambda}^{n}f(z)}\right]\geq\alpha\left|\frac{D_{\lambda}^{n+m}f(z)}{D_{\lambda}^{n}f(z)}-1\right|,\ z\in E.$$

equivalently

$$\operatorname{Re}\left[\frac{z^{p}-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}{z^{p}-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}\right]\geq\alpha\left|\frac{z^{p}-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}{z^{p}-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}-1\right|$$

$$=\operatorname{Re}\left[\frac{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}\right]\geq\alpha\left|\frac{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}-1\right|$$

$$=\alpha\left|\frac{\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}\right| \qquad (3)$$

Choosing values of z on the real so that left side of (3) is real and letting $z\to 1$, we get

$$\left[1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}\right]\geq\alpha\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right]a_{k+p-1}$$

$$\Rightarrow\alpha\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right]a_{k+p-1}+\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}\leq 1$$

which yields

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^{m}-\alpha\right]a_{k+p-1}\leq 1$$

Conversely, suppose that (2) is true for $z\in E$, then

$$\operatorname{Re}\left[\frac{D_{\lambda}^{n+m}f(z)}{D_{\lambda}^{n}f(z)}\right]\geq\alpha\left|\frac{D_{\lambda}^{n+m}f(z)}{D_{\lambda}^{n}f(z)}-1\right|\geq 0$$

$$\left[1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}a_{k+p-1}\right]\geq\alpha\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\right]\right]a_{k+p-1}$$

If

$$\left[\frac{1-\sum_{n=2}^{\infty}\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\right]a_{k+p-1}|z|^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}a_{k+p-1}|z|^{k-1}}\right]-\alpha\frac{\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[[1+(k+p-2)\lambda]^{m}-1\right]a_{k+p-1}|z|^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}a_{k+p-1}|z|^{k-1}}\geq 0$$

that is, if

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)(1+(k+p-2)\lambda)^{m}-\alpha\right]a_{k+p-1}\leq 1$$

This completes the proof.

Corollary 2.1. Let the function $f(z)$ defined by (1) is in the class $T(n, m, j, \alpha, \lambda)$ then

$$0\leq \alpha_k\leq\frac{1}{\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)(1+(k+p-2)\lambda)^{m}-\alpha\right]},k\geq 2$$

This result is sharp for the function

$$f(z)=z-\frac{z^{k-i}}{\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)(1+(k+p-2)\lambda)^{m}-\alpha\right]} \tag{4}$$

Theorem 2.2. Let the function $0\geq\alpha_1\leq\alpha_2$, then $T(n, m, j, \alpha_1, \lambda)\supseteq T(n, m, j, \alpha_2, \lambda)$.

Proof. Let the function $f(z)$ defined by (1) be in the class $f(z)\in T(n, m, j, \alpha_1, \lambda)$ then by theorem (2.1),

Assume that $f(z)\in T(n, m, j, \alpha_1, \lambda)$, then by definition

$$\mathrm{Re}\left[\frac{D_{\lambda}^{n+m}f(z)}{D_{\lambda}^{n}f(z)}\right]\geq\alpha_1\left|\frac{D_{\lambda}^{n+m}f(z)}{D_{\lambda}^{n}f(z)}-1\right|,\ z\in E.$$

equivalently

$$\mathrm{Re}\left[\frac{z^p-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}{z^p-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}\right]\geq\alpha_1\left|\frac{z^p-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}{z^p-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}-1\right|$$

$$=\mathrm{Re}\left[\frac{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}\right]\geq\alpha_1\left|\frac{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}-1\right|$$

$$=\alpha_1\left|\frac{\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}\right| \tag{5}$$

Choosing values of z on the real so that left side of (5) is real and letting $z\to 1$, we get

$$\left[1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}\right]\geq \alpha_1\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right]a_{k+p-1}$$

$$\Rightarrow \alpha_1\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right]a_{k+p-1}+\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}\leq 1$$

which yields

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha_1+1)\,[1+(k+p-2)\lambda]^{m}-\alpha_1\right]a_{k+p-1}\leq 1$$

and for $f(z)\in T(n,\, m,\, j,\, \alpha_2,\, \lambda)$, then by definition

$$\operatorname{Re}\left[\frac{D_{\lambda}^{n+m}f(z)}{D_{\lambda}^{n}f(z)}\right]\geq \alpha_2\left|\frac{D_{\lambda}^{n+m}f(z)}{D_{\lambda}^{n}f(z)}-1\right|,\ z\in E.$$

equivalently

$$\operatorname{Re}\left[\frac{z^p-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}{z^p-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}\right]\geq \alpha_2\left|\frac{z^p-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}{z^p-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k+p-1}}-1\right|$$

$$=\operatorname{Re}\left[\frac{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}\right]\geq \alpha_2\left|\frac{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}-1\right|$$

$$=\alpha_2\left|\frac{\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}\right| \qquad (6)$$

Choosing values of z on the real so that left side of (6) is real and letting $z\to 1,$ we get

$$\left[1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}\right]\geq \alpha_2\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right]a_{k+p-1}$$

$$\Rightarrow \alpha_2\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right]a_{k+p-1}+\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}\leq 1$$

which yields

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha_2+1)\,[1+(k+p-2)\lambda]^{m}-\alpha_2\right]a_{k+p-1}\leq 1$$

Now we have

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha_2+1)\,[1+(k+p-2)\lambda]^{m}-\alpha_2\right]a_{k+p-1}\leq 1$$

Consequently

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha_1+1)\,[1+(k+p-2)\lambda]^{m}-\alpha_1\right]a_{k+p-1}$$

$$\leq \sum_{k=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]} \left[(\alpha_2+1) [1+(k+p-2)\lambda]^m - \alpha_2 \right] a_{k+p-1}.$$

This completes the proof of theorem (2.2) with the help of theorem (2.1)

Theorem 2.3 For $\alpha \geq 0,\ T(n+1, m, j, \alpha, \lambda) \subseteq T(n, m, j, \alpha, \lambda)$.

Proof. Let the function $f(z)$ defined by (1) be in the class $T(n+1, m, j, \alpha, \lambda)$ then by theorem (2.1),

Assume that $f(z) \in T(n+1, m, j, \alpha, \lambda)$, then by definition

$$\operatorname{Re}\left[\frac{D_\lambda^{n+m} f(z)}{D_\lambda^{n+1} f(z)}\right] \geq \alpha \left|\frac{D_\lambda^{n+m} f(z)}{D_\lambda^{n+1} f(z)} - 1\right|,\ z \in E.$$

equivalently

$$\operatorname{Re}\left[\frac{z^p - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}}{z^p - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}}\right] \geq \alpha \left|\frac{z^p - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}}{z^p - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}} - 1\right|$$

$$= \operatorname{Re}\left[\frac{1 - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k-1}}{1 - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k-1}}\right] \geq \alpha \left|\frac{1 - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k-1}}{1 - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k-1}} - 1\right|$$

$$= \alpha \left|\frac{\sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k-1} - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k-1}}{1 - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k-1}}\right| \qquad (7)$$

Choosing values of z on the real so that left side of (7) is real and letting $z \to 1$, we get

$$\left[1 - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1}\right] \geq \alpha \sum_{n=2}^{\infty} \left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right] - \left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right] a_{k+p-1}$$

$$\Rightarrow \alpha \sum_{n=2}^{\infty} \left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right] - \left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right] a_{k+p-1} + \sum_{k=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1} \leq 1$$

which yields

$$\sum_{k=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]} \left[(\alpha+1) [1+(k+p-2)\lambda]^m - \alpha\right] a_{k+p-1} \leq 1$$

and for class $T(n, m, j, \alpha, \lambda)$

Assume that $f(z) \in T(n, m, j, \alpha, \lambda)$, then by definition

$$\operatorname{Re}\left[\frac{D_\lambda^{n+m} f(z)}{D_\lambda^{n} f(z)}\right] \geq \alpha \left|\frac{D_\lambda^{n+m} f(z)}{D_\lambda^{n} f(z)} - 1\right|,\ z \in E.$$

equivalently

$$\operatorname{Re}\left[\frac{z^p - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}}{z^p - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}}\right] \geq \alpha \left|\frac{z^p - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}}{z^p - \sum_{n=2}^{\infty} \frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)} a_{k+p-1} z^{k+p-1}} - 1\right|$$

$$= \operatorname{Re}\left[\frac{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}\right] \geq \alpha\left|\frac{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}-1\right|$$

$$=\alpha\left|\frac{\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}{1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}a_{k+p-1}z^{k-1}}\right| \tag{8}$$

Choosing values of z on the real so that left side of (8) is real and letting $z \to 1,$ we get

$$\left[1-\sum_{n=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1}\right] \geq \alpha\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right]a_{k+p-1}$$

$$\Rightarrow \alpha\sum_{n=2}^{\infty}\left[\left[\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}\right]-\left[\frac{[1+(k+p-2)\lambda]^{n}}{(1-\lambda+\lambda p)}\right]\right]a_{k+p-1}+\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{(1-\lambda+\lambda p)}a_{k+p-1} \leq 1$$

which yields

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^{m}-\alpha\right]a_{k+p-1} \leq 1 .$$

Now we have

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^{m}-\alpha\right]a_{k+p-1} \leq 1 .$$

Consequently

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^{m}-\alpha\right]a_{k+p-1}$$

$$\leq \sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+m}}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^{m}-\alpha\right]a_{k+p-1}$$

This completes the proof of theorem (2.3) with the help of theorem (2.1)

Theorem 2.4

$T(n, m, j, \alpha, \lambda)$ is a convex set

Proof. Let the function

$$f(z)=z^{p}-\sum_{n=2}^{\infty}a_{k+p-1,\nu}z^{k+p-1},\ (a_{k},\nu \geq 0;\nu=1,2) \tag{9}$$

be in the class $T(n, m, j, \alpha, \lambda)$. It is sufficient to show that the function $h(z)$ defined by

$h(z)=\mu f_{1}(z)+(1-\mu)f_{2}(z), \quad (0\leq\mu\leq1)$

Is also in the class $T(n, m, j, \alpha, \lambda)$. Since $0\leq\mu\leq1$

$$h(z)=z^{p}-\sum_{n=2}^{\infty}\left[\mu\, a_{k+p-1,1}+(1-\mu)\,a_{k+p-1,2}\right]z^{k+p-1}$$

With the aid of theorem 2.1, we obtain

$$\leq\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n+1}}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^{m}-\alpha\right]\left[\mu\, a_{k+p-1,1}+(1-\mu)\,a_{k+p-1,2}\right]\leq1$$

which implies that $h(z)\in T(n, m, j, \alpha, \lambda)$

hence $T(n, m, j, \alpha, \lambda)$ is a convex set

Theorem 2.5. Let the function $f(z)$ defined by (1), be in the class $T(n, m, j, \alpha, \lambda)$ then for $|z| = r < 1$

$$\left|D_\lambda^i\right| \geq r - \frac{r^2}{(1+\lambda)^{n-i}\left[(\alpha+1)(1+\lambda)^m - \alpha\right]} \tag{10}$$

and

$$\left|D_\lambda^i\right| \leq r + \frac{r^2}{(1+\lambda)^{n-i}\left[(\alpha+1)(1+\lambda)^m - \alpha\right]} \tag{11}$$

for $z \in E$ and $0 \leq i \leq n$.

Proof. Note that $f(z) \in T(n, m, j, \alpha, \lambda)$. if and only if $D_\lambda^i \in T(n, m, j, \alpha, \lambda)$ and that

$$D_\lambda^i = z^p - \sum_{n=2}^{\infty}\left[\frac{[1+(k+p-2)\lambda]}{[1-\lambda+\lambda p]}\right] a_{k+p-1} z^{k+p-1}. \tag{12}$$

By theorem (2.1) we know that

$$(1+\lambda)^{n-i}\left[(\alpha+1)[1+\lambda]^m - \alpha\right]\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^i}{[1-\lambda+\lambda p]} a_{k+p-1}$$

$$\leq \sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right] a_{k+p-1} \leq 1$$

that is

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^i}{[1-\lambda+\lambda p]} a_{k+p-1} \leq \frac{1}{(1+\lambda)^{n-i}\left[(\alpha+1)(1+\lambda)^m - \alpha\right]} \tag{13}$$

$$\left|D_\lambda^i f(z)\right| \leq |z| + r^2\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^i}{[1-\lambda+\lambda p]} a_{k+p-1} \leq r + r^2\frac{1}{(1+\lambda)^{n-i}\left[(\alpha+1)(1+\lambda)^m - \alpha\right]}$$

and

$$\left|D_\lambda^i f(z)\right| \geq r - r^2\frac{1}{(1+\lambda)^{n-i}\left[(\alpha+1)(1+\lambda)^m - \alpha\right]}$$

This completes the proof of theorem (2.5)

Corollary 2.2. Let the function $f(z)$ defined by (1) be in the class $T(n, m, j, \alpha, \lambda)$ then for $|z| = r < 1$

$$|f(z)| \geq r - \frac{r^2}{(1+\lambda)^{n-i}\left[(\alpha+1)(1+\lambda)^m - \alpha\right]} \tag{14}$$

and

$$|f(z)| \leq r + \frac{r^2}{(1+\lambda)^{n-i}\left[(\alpha+1)(1+\lambda)^m - \alpha\right]}, z \in E \tag{15}$$

The inequalities in (14) and (15) are attained for the function given by

$$f(z) = z - \frac{z^2}{(\alpha+1)(1+\lambda)^m - \alpha}$$

Proof. Taking $i = 0$ in theorem (2.5), we immediately obtained (14) and (15).

Theorem 2.6 Let $f_j(0) = z$ and

$$f_k(z)=z^p-\frac{1}{\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]}z^{k+p-1},\ (k\geq 2;n\in N)$$

For $\alpha\geq 0$. then $f(z)$ is in the class $T(n,\ m,\ j,\ \alpha,\ \lambda)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=2}^{\infty}\mu_k f_k(z) \text{ where } \mu_k\geq 0 \text{ and } \sum_{n=2}^{\infty}\mu_k=1 \tag{16}$$

Proof. Assume that

$$\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]\frac{1}{\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]}\mu_k$$

$$=\sum_{n=2}^{\infty}\mu_k=1-\mu_j\leq 1$$

So by Theorem (2.1), $f(z)\in T(n,\ m,\ j,\ \alpha,\ \lambda)$

Conversely, assume that the function defined by (1) belongs to class. Then

$$a_{k+p-1}\leq\frac{1}{\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]},$$ Then it follows that

$$f_k(z)=\sum_{n=2}^{\infty}\mu_k f_k(z)=z^p-\sum_{k=2}^{\infty}\frac{1}{\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]}\mu_k z^{k+p-1}$$

$(k\geq 2,\ n\in N_0)$.

Setting

$$\mu_k=\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]a_{k+p-1},$$

and $\mu_j=1-\sum_{n=2}^{\infty}\mu_k$.

we can see that $f(z)$ can be expressed in the form (16). This completes the proof of Theorem (2.6)

Theorem 2.7 Let the function $f(z)$ defined by (1), be in the class $T(n,\ m,\ j,\ \alpha,\ \lambda)$ then $f(z)$ is close to convex of order $\rho(0\leq\rho<1)$ in $|z|<r_1$, where

$$r_1=r_1(n,\ m,\ j,\ \alpha,\ \rho)$$

$$=\inf_k\left[\left(\frac{1-\rho}{k}\right)\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]\right]^{\frac{1}{k-1}} \tag{17}$$

The result is sharp with the extremal function $f(z)$ given by (4).

Proof. We must show that $|f'(z)-1|\leq(1-\rho)$ for $|z|<r_1(n,\ m,\ j,\ \alpha,\ \rho)$ where $r_1(n,\ m,\ j,\ \alpha,\ \rho)$ is given by (17). Indeed we find form (1) that

$$|f'(z)-1|\leq\sum_{k=2}^{\infty}k\,a_k|z|^{k+p-2}$$

Thus $|f'(z)-1|\leq(1-\rho)$,

$$i\,f\sum_{k=2}^{\infty}\frac{k}{(1-\rho)}a_k|z|^{k+p-2}\leq 1. \tag{18}$$

But by Theorem (2.1) , (18) will be true if

$$\frac{k}{(1-\rho)}|z|^{k-2}\leq\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]$$

That is, if

$$|z|\leq\left[\left(\frac{1-\rho}{k}\right)\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]\right]^{\frac{1}{k-1}} \tag{19}$$

Theorem (2.7) follows from (19).

Theorem 2.8. Let the function $f(z)$ defined by (1), be in the class $T(n,\, m,\, j,\, \alpha,\, \lambda)$ then $f(z)$ is starlike of order $\rho(0\leq\rho<1)$ in $|z|<r_2$, where

$$r_2=r_2(n,\, m,\, j,\, \alpha,\, \rho)$$

$$=\inf_k\left[\left(\frac{1-\rho}{k-\rho}\right)\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]\right]^{\frac{1}{k-1}} \tag{20}$$

The result is sharp with the extremal function $f(z)$ given by (4).

Proof. It suffices to show that

$\left|\dfrac{z\,f'(z)}{f'(z)-1}\right|\leq(1-\rho)$ for $|z|<r_2(n,\, m,\, j,\, \alpha,\, \rho)$ where $r_2(n,\, m,\, j,\, \alpha,\, \rho)$ is given by (20). Indeed we find form (1) that

$$\sum_{k=2}^{\infty}\frac{k-\rho}{(1-\rho)}a_{k+p-1}|z|^{k+p-2}\leq 1.$$

(21)

But by Theorem (2.1), (21) will be true if

$$\frac{(1-\rho)}{(k-\rho)}|z|^{k-2}\leq\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]$$

That is, if

$$|z|\leq\left[\left(\frac{1-\rho}{k-\rho}\right)\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]\right]^{\frac{1}{k-1}} \tag{22}$$

Theorem (2.8) follows easily from (22).

Theorem 2.9. Let the function $f(z)$ defined by (1), be in the class $T(n,\, m,\, j,\, \alpha,\, \lambda)$ then $f(z)$ is convex of order $\rho(0\leq\rho<1)$ in $|z|<r_3$, where

$$r_3=r_3(n,\, m,\, j,\, \alpha,\, \rho)$$

$$=\inf_k\left[\left(\frac{1-\rho}{k\,(k-\rho)}\right)\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\left(1+(k+p-2)\lambda\right)^m-\alpha\right]\right]^{\frac{1}{k-1}} \tag{23}$$

The result is sharp with the extremal function $f(z)$ given by (4).

Proof. It is suffices to show that

$\left|\frac{z\,f'(z)}{z\,f'(z)-1}\right| \le (1-\rho)$ for $|z| < r_3(n, m, j, \alpha, \rho)$ where $r_3(n, m, j, \alpha, \rho)$ is given by (23). Indeed we find form (1) that

$$\sum_{k=2}^{\infty} \frac{k(k-\rho)}{(1-\rho)} a_{k+p-1} |z|^{k+p-2} \le 1.$$

(24)

But by Theorem (2.1), (24) will be true if

$$\frac{(1-\rho)}{k(k-\rho)}|z|^{k-2} \le \frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)(1+(k+p-2)\lambda)^m - \alpha\right]$$

That is, if

$$|z| \le \left[\left(\frac{1-\rho}{k(k-\rho)}\right)\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)(1+(k+p-2)\lambda)^m - \alpha\right]\right]^{\frac{1}{k-1}} \quad (25)$$

Theorem (2.9) follows easily from (25).

Theorem 2.10. Let the function $f(z)$ defined by (1), be in the class $T(n, m, j, \alpha, \lambda)$ and let c be a real number such that $c > -1$. then the function $F(z)$ defined by

$$F(z) = z^p - \int_0^z t^{c-1} f(t)\,dt, \quad (c > -1) \quad (26)$$

also belongs to the class $T(n, m, j, \alpha, \lambda)$.

Proof. From the representation of (26) of $f(z)$ it follows that

$$F(z) = z^p - \sum_{n=2}^{\infty} b_{k+p-1} z^{k+p-1}, \quad where \quad b_{k+p-1} = \frac{c+1}{c+k} a_{k+p-1}$$

Therefore, we have

$$\sum_{k=2}^{\infty} \frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right] b_{k+p-1}$$

$$\sum_{k=2}^{\infty} \frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right] a_{k+p-1}$$

≤ 1

Since $f(z) \in T(n, m, j, \alpha, \lambda)$.

Hence by theorem (2.1), $F(z) \in T(n, m, j, \alpha, \lambda)$.

Theorem 2.11. Let the function $f(z)$ defined by (1), be in the class $T(n, m, j, \alpha, \lambda)$ and let c be a real number such that $c > -1$. then the function $F(z)$ defined by (26) is multivalent in $|z| < R^*$ where

$$R^* = \inf_k \left[\left(\frac{(c+1)\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)(1+(k+p-2)\lambda)^m - \alpha\right]}{(c+k)}\right)\right]^{\frac{1}{k-1}}, k \ge 2 \quad (27)$$

The result is sharp.

Proof. From (26), we have

$$f(z)=\frac{z^{(1-c)}\left[z^{c}F(z)\right]'}{c+1}=z^{p}-\sum_{n=2}^{\infty}\frac{c+k}{c+1}a_{k+p-1}z^{k+p-1}$$

In order to obtain the required result, it suffices to show that
$|F'(z)-1|<1,$ whenever $|z|<R^{*}$ where R^{*} is given by (27).

Now $|F'(z)-1|<1$ if $\sum_{n=2}^{\infty}\frac{k\,(c+k)}{c+1}a_{k+p-1}|z|^{k+p-2}\le 1$ (28)

But by Theorem 2.1 confirm that

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^{m}-\alpha\right]\le 1 \quad (29)$$

Thus

$$\frac{k\,(c+k)}{c+1}|z|^{k+p-2}<\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^{m}-\alpha\right]$$

Thus

$$|z|<\left[\frac{k\,(c+k)}{c+1}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^{m}-\alpha\right]\right]^{\frac{1}{k-1}},\ k\ge 2 \quad (30)$$

Therefore, the function given by (26) is multivalent in $|z|<R^{*}$.

Let the function $f_{\nu}(z),(\nu=1,2)$ be defined by (9). The modified Hadmard product $f_{1}(z)$ and $f_{2}(z)$ is defined by

$$(f_{1}*f_{2})(z)=z^{p}-\sum_{n=2}^{\infty}a_{k,1}\,a_{k,2}\,z^{k+p-1}$$

(31)

Theorem 2.12. Let each of the function $f_{\nu}(z),(\nu=1,2)$ be defined by (9) be in the class $T(n,m,j,\alpha,\lambda)$, then $f_{1}*f_{2}(z)\in T(n,m,j,\alpha,\lambda)$ where

$$\beta=\frac{(1+\lambda)^{n}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^{m}-\alpha\right]^{2}-(1+\lambda)^{m}}{(1+\lambda)^{m}-1}$$

The result is sharp.

Proof. Employing the technique used by Schild and Silverman [10], we need to find largest $\beta=\beta(n,m,j,\alpha,\lambda)$ such that

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\beta+1)\,[1+(k+p-2)\lambda]^{m}-\beta\right]a_{k,1}a_{k,2}\le 1$$

Since $\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^{m}-\alpha\right]a_{k,1}\le 1$

and $\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^{m}-\alpha\right]a_{k,2}\le 1$

By the Cauchy – Schwarz inequality, we have

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^{m}-\alpha\right]\sqrt{a_{k,1}a_{k,2}}\le 1$$

and thus it is sufficient to show that

$$\sum_{k=2}^{\infty}\frac{[1+(k+p-2)\lambda]^{n}}{[1-\lambda+\lambda p]}\left[(\beta+1)\,[1+(k+p-2)\lambda]^{m}-\beta\right]a_{k,1}a_{k,2}\le$$

$$\le \sum_{k=2}^{\infty} \frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]} \left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right] \sqrt{a_{k,1}a_{k,2}} \le 1$$

That is $\sqrt{a_{k,1}a_{k,2}} \le \sum_{k=2}^{\infty} \frac{\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right]}{\left[(\beta+1)[1+(k+p-2)\lambda]^m - \beta\right]}$, $(k \ge 2)$.

Note that $\sqrt{a_{k,1}a_{k,2}} \le \dfrac{1}{\dfrac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right]}$

Consequently, we need only prove that

$$\frac{1}{\dfrac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right]} \le \frac{\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right]}{\left[(\beta+1)[1+(k+p-2)\lambda]^m - \beta\right]}$$

Or, equivalently that

$$\beta\left[[1+(k+p-2)\lambda]^m - 1\right] + [1+(k+p-2)\lambda]^m \le \frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right]^2$$

$$\beta \le \frac{\dfrac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right]^2 - [1+(k+p-2)\lambda]^m}{\left[[1+(k+p-2)\lambda]^m - 1\right]} \qquad (33)$$

Since right hand side of (33) is an increasing function of k, letting $k=2$ and $p=1$ in equation (33), we have

$$\beta \le \frac{[1+\lambda]^n\left[(\alpha+1)[1+\lambda]^m - \alpha\right]^2 - [1+\lambda]^m}{[1+\lambda]^m - 1}$$

Which proves the main assertion of theorem (2.12). finally, by taking the function

$$f_\nu(z) = z^p - \frac{1}{[1+\lambda]^n\left[(\alpha+1)(1+\lambda)^m - \alpha\right]} z^2 \qquad (34)$$

We can see the result is sharp.

Theorem 2.13. Let the function $f_\nu(z), (\nu = 1, 2)$ defined by (9) be in the class $T(n, m, j, \alpha, \lambda)$, then the function

$$h(z) = z^p - \sum_{n=2}^{\infty} \left(a_{k+p-1,1}^2 + a_{k+p-1,2}^2\right) z^{k+p-1} \qquad (35)$$

belongs to class $T(n, m, j, \alpha, \lambda)$, where

$$\eta = \eta(n, m, j, \alpha, \lambda) = \frac{[1+\lambda]^n\left[(\alpha+1)[1+\lambda]^m - \alpha\right]^2 - [1+\lambda]^m}{2[1+\lambda]^m - 1} \qquad (36)$$

The result is sharp for the function defined by (34).

Proof. By virtue of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right] a_{k,1}^2 \le$$

$$\le \sum_{k=2}^{\infty}\left[\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right] a_{k,1}\right]^2 \le 1 \qquad (37)$$

and $\sum_{k=2}^{\infty} \dfrac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)[1+(k+p-2)\lambda]^m - \alpha\right] a_{k+p-1,2}^2 \le 1$

$$\leq \sum_{k=2}^{\infty}\left[\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^m-\alpha\right]a_{k+p-1,2}\right]^2 \leq 1 \qquad (38)$$

it follows from (37) and (38)

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^m-\alpha\right]\right]^2\left(a_{k+p-1,1}^2+a_{k+p-1,2}^2\right)\leq 1 \qquad (39)$$

Therefore, we need to find the largest η such that

$$\left[\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\eta+1)\,[1+(k+p-2)\lambda]^m-\eta\right]\right]$$

$$\leq \frac{1}{2}\left[\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^m-\alpha\right]\right]^2$$

That is

$$\eta \leq \frac{\left[\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^m-\alpha\right]\right]^2-2\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[(\alpha+1)\,[1+(k+p-2)\lambda]^m\right]}{2\frac{[1+(k+p-2)\lambda]^n}{[1-\lambda+\lambda p]}\left[\,[1+(k+p-2)\lambda]^m-1\right]} \qquad (40)$$

Since ritht hand side of (40) is an increasing function of k, we readily have

$$\eta \leq \frac{\left[[1+(k+p-2)\lambda]^n\left[(\alpha+1)\,[1+(k+p-2)\lambda]^m-\alpha\right]\right]^2-2[1+(k+p-2)\lambda]^m}{2\left[\,[1+(k+p-2)\lambda]^m-1\right]}$$

and Theorem(2.13) follows at once .

REFERENCES

[1] Al-Oboudi, F.M.,(2004), On univalent functions defined by a generlized Salagean operator, Int. J. Math. Math. Sci., 25-28, 1429-1436.

[2] Goodman, A.W., (1991), On uniformly convex functions , Ann.Polon.Math.56(1991), 87-92

[3] Goodman, A.W.,(1991), On uniformly starlike functions , J.Math.Anal.Appl.155, 364370

[4] K.K. Dixit, Saurabh Porwal and Sanjay Kumar Ghai, On subclasses of Multivalent functions with negative coefficients defined by Generalized Salagean operator, Journal of Rajasthan Academy of Physical Sciences(2013),pp 151-166

[5] Murugusundaramoorthy, G.(1994), Studies on Classes of Analytc function with Negative Coe cient , Ph.D.Thesis, University of Madras, India.

[6] Pathak,A.L.(2004), A Study of Univalent and Related Functions , Ph.D.Thesis, C.S.J.M. University, Kanpur, India.

[7] Robertson,M.S. (1936), On the theory of Univalent Functions , Annals Math., 37,374408.

[8] Ronning,F. (1993), Uniformly convex Functions and a corresponding class of Starlike Functions, Proc.Amer.Math.Soc.118(1),189-196.

[9] Salagean, G.S., (1983), Subclass of univalent functions, Lecture Notes in Math.SpringerVerlag, 1013, pp.362-372.

[10] Schild,A.and Silverman,H.(1975),Convolution of Univalent function with Negative Coe cient , Ann. Univ.Marie-Curie-Sklodowska 29,99-107

[11] Silverman,H. (1975), Univalent function with Negative Coe cient , Proc.Amer.Math.Soc.51, 109-116.