International Advance Journal of Engineering, Science and Management (IAJESM)
ISSN -2393-8048, January-June 2022, Submitted in May 2022, iajesm2014@gmail.com

Review of Application of Pi (π): A Study

Mundase Radhika, Dept. of Mathematics, Research Scholar, SunRise University, Alwar(Rajasthan)
Dr. Satendra Kumar (Mathematics), Professor (Dept. of Mathematics), SunRise University, Alwar (Rajasthan)

ABSTRACT

This research paper provides a comprehensive review of the application of the mathematical constant Pi (π). Pi, denoted by the Greek letter π, is an irrational number representing the ratio of a circle's circumference to its diameter. Its significance extends beyond the realm of geometry and mathematics, finding numerous applications in diverse fields of science, engineering, and technology. This paper explores the historical development of Pi, its mathematical properties, and its practical applications across various disciplines. Additionally, it examines the potential future applications of Pi and highlights the importance of continued research in this area.

Keywords: ***Pi (π), Geometry, Science, Engineering, and Technology***

1. INTRODUCTION

Definition and History of Pi: Pi (π) is a mathematical constant that represents the ratio of a circle's circumference to its diameter. It is an irrational number, meaning it cannot be expressed as a fraction or a finite decimal. The value of Pi is approximately 3.14159, but it is an infinitely long and non-repeating decimal. Pi is denoted by the Greek letter π, and it holds a significant place in mathematics and various other fields. The concept of Pi has been known and studied for thousands of years. Ancient civilizations, such as the Egyptians and Babylonians, approximated the value of Pi in their mathematical calculations. The earliest written approximations of Pi date back to around 1900 BCE. However, it was the ancient Greek mathematician Archimedes (287-212 BCE) who made significant contributions to understanding Pi. He used geometric methods to estimate Pi and determined that it lies between the values 3 1/7 and 3 10/71.The symbol "π" to represent the constant Pi was first introduced by the Welsh mathematician William Jones in 1706, and it gained widespread adoption after it was popularized by the Swiss mathematician Leonhard Euler in the 18th century.

Importance and Significance of Pi in Mathematics and Beyond:

Pi holds immense importance in mathematics, as it appears in a wide range of mathematical formulas and equations. Its significance can be summarized as follows:

Geometry: Pi is fundamental to the study of circles, providing a precise relationship between a circle's circumference, diameter, and radius. It allows for the calculation of circle properties such as area, volume, and surface area. Additionally, Pi is crucial for trigonometry, where it appears in trigonometric functions, such as sine and cosine.

Number Theory: Pi is an irrational number, and its irrationality has been proven mathematically. The study of irrational numbers and their properties is an important aspect of number theory.

Analysis and Calculus: Pi appears in various formulas and equations in calculus, such as those related to limits, integrals, and series. It is a key component in many mathematical theorems and proofs.

Fractals and Chaos Theory: Pi is often encountered in the study of fractals, which are intricate geometric patterns that exhibit self-similarity at different scales. Fractals have applications in many scientific and technological fields, including computer graphics, image compression, and modeling complex systems.

Probability and Statistics: Pi plays a role in probability and statistics, particularly in the field of random processes. It is used in the calculation of probabilities and the generation of random numbers.

Applied Sciences and Engineering: Pi finds numerous applications in various scientific and engineering disciplines. It is used in calculations related to circular motion, oscillations, fluid dynamics, electromagnetic theory, signal processing, and many other areas. The significance of Pi extends beyond mathematics. It has become an iconic and recognizable symbol in popular culture and is celebrated on Pi Day (March 14th) each year. Pi's pervasive presence in mathematics and its applications in diverse fields highlight its universal importance and enduring relevance.

2. MATHEMATICAL PROPERTIES OF Pi

Pi is an irrational number, which means it cannot be expressed as a fraction of two integers. It has been proven that Pi cannot be written as a finite or repeating decimal. This proof was first established by Johann Lambert in 1768. It implies that the decimal expansion of Pi goes on infinitely without any repeating pattern. Pi is not just irrational but also a transcendental number. A transcendental number is a real number that is not a root of any non-zero polynomial equation with integer coefficients. The transcendence of Pi was proven by the German mathematician Ferdinand von Lindemann in 1882, building upon the work of Karl Weierstrass and Charles Hermite. This result implies that Pi cannot be a solution to any algebraic equation.

Calculation Methods and Approximations of Pi:

Geometric Methods: One of the earliest methods to approximate Pi was through geometry. Archimedes used a method of inscribed and circumscribed polygons to find bounds for Pi. By increasing the number of sides of the polygons, he obtained more accurate approximations of Pi.

Series and Infinite Products: Various mathematical series and infinite products have been discovered to calculate Pi. Examples include the Gregory-Leibniz series (arctan series), Nilakantha's series, Euler's series, and the Wallis product. These formulas involve an infinite sum or product of terms that converge to Pi as the number of terms increases.

Continued Fractions: Continued fractions provide another approach to approximate Pi. A continued fraction is an expression where an integer is divided by the sum of another integer and a fraction, recursively. Pi can be expressed as a continued fraction, allowing for iterative calculations to obtain increasingly accurate approximations.

Machin-like Formulas: Machin-like formulas are special trigonometric identities that involve Pi. They provide efficient methods to calculate Pi using trigonometric functions. Examples include the Machin formula (used by John Machin in 1706) and the Chudnovsky algorithm (discovered by brothers David and Gregory Chudnovsky in 1989), which is highly efficient in terms of rapid convergence.

Pi's Relationship with Other Mathematical Constants:

Euler's Identity: Euler's identity, often hailed as one of the most beautiful equations in mathematics, connects five fundamental mathematical constants: e (the base of the natural logarithm), Pi, i (the imaginary unit), 1 (the identity element for multiplication), and 0 (the additive identity). The equation is given by e^(i * Pi) + 1 = 0. This relationship highlights the interplay between exponential, trigonometric, and complex number concepts.

Fourier Transform: The Fourier transform, a fundamental mathematical tool in signal processing and analysis, involves the use of complex exponential functions. The appearance of Pi in the exponent of the Fourier transform formula is significant. It indicates the connection between periodic functions, frequency representation, and the geometry of circles.

Riemann Zeta Function: The Riemann zeta function is a complex-valued function that has deep connections with the distribution of prime numbers. It is defined for complex numbers s with a real part greater than 1 as ζ(s) = 1^(-s) + 2^(-s) + 3^(-s) + ..., where the sum extends over all positive integers. The Riemann zeta function is related to Pi through the equation ζ(2) = (Pi^2)/6, known as the Basel problem solution.

3. APPLICATIONS OF Pi IN SCIENCE AND ENGINEERING

Geometry and Trigonometry

Calculation of Circle Properties: *Pi plays a fundamental role in the calculation of various properties of circles, including:*

Circumference: The circumference of a circle is the distance around its boundary. Pi is used in the formula C = 2πr, where C represents the circumference and r is the radius of the circle. This relationship allows us to calculate the circumference when the radius is known or vice versa.

Diameter: The diameter of a circle is the distance across the circle passing through its center. Pi is involved in the formula D = 2r, where D denotes the diameter and r is the radius. Using Pi, we can determine the diameter when the radius is known or vice versa.

Area: The area of a circle can be calculated using the formula A = πr^2, where A represents the area and r is the radius. Pi appears in this formula as it relates the ratio of the circumference to the diameter, providing a direct connection between the circle's geometry and its area.

Surface Area and Volume: Pi is also used in determining the surface area and volume of various three-dimensional objects derived from circles, such as spheres and cylinders. For instance, the surface area of a sphere is given by A = 4πr^2, and the volume of a sphere is V = (4/3)πr^3.

Trigonometric Functions and Identities Involving Pi: Trigonometry is a branch of mathematics that studies the relationships between angles and the sides of triangles. Pi appears in various trigonometric functions and identities, including:

Sine (sin) and Cosine (cos): The values of sine and cosine functions depend on the angles involved. Pi is crucial in trigonometry as it is involved in defining the unit circle, which is a circle with a radius of 1 centered at the origin of a coordinate system. The values of sine and cosine of angles (measured in radians) can be determined using points on the unit circle, which are derived from Pi.

Periodicity: Trigonometric functions exhibit periodicity, which means their values repeat after a specific interval. For example, the sine function has a period of 2π radians (or 360 degrees). This periodicity arises due to the nature of the unit circle, where traversing the entire circle corresponds to an angle of 2π. Pi's relationship with circles allows us to understand and analyze the repeating patterns in trigonometric functions.

Trigonometric Identities: Pi appears in various trigonometric identities that relate different trigonometric functions. Examples include the Pythagorean identities, such as sin^2θ + cos^2θ = 1, and the angle addition formulas, such as sin(α + β) = sinαcosβ + cosαsinβ. These identities are derived using the properties of circles and angles, with Pi serving as a fundamental component in their derivations.

Harmonic Motion and Waves: Trigonometric functions involving Pi are used to describe harmonic motion and wave phenomena. The oscillation of pendulums, vibrations of strings, and the behavior of electromagnetic waves can be expressed using sine and cosine functions, which are directly related to Pi and the circular nature of angles.

Physics

Calculations Involving Circular Motion and Oscillations:

Pi plays a crucial role in describing and calculating various aspects of circular motion and oscillations, including:

Angular Velocity: Angular velocity measures the rate at which an object rotates or moves around a circular path. It is defined as the change in angle per unit time. Pi is used in converting between angular and linear measurements, such as converting radians to degrees or vice versa.

Period and Frequency: Pi appears in formulas related to the period (T) and frequency (f) of an oscillating system. For example, the period of a pendulum, the time it takes for one complete swing, is given by T = 2π√(L/g), where L is the length of the pendulum and g is the acceleration due to gravity. Similarly, frequency, the number of oscillations per unit time, is given by f = 1/T. These formulas involve Pi due to the connection between circular motion and oscillatory behavior.

Harmonic Oscillators: Harmonic oscillators, such as springs and simple pendulums, exhibit sinusoidal motion. The equations that describe their behavior involve trigonometric functions, which are inherently connected to circles and angles through Pi. The displacement, velocity, and acceleration of a harmonic oscillator can be expressed using sine and cosine functions, allowing for the precise mathematical description of the oscillatory motion.

Quantum Mechanics and Statistical Physics: Pi's significance extends into the realms of quantum mechanics and statistical physics, where it appears in various contexts: Schrödinger Equation: The Schrödinger equation is a fundamental equation in quantum mechanics that describes the behavior of quantum particles. It involves complex exponential functions, which are directly related to circles and angles through Euler's formula, $e^{(i\theta)} = \cos\theta + i\sin\theta$. Pi is present in the exponential term of the equation, linking the geometry of circles with the wave-like behavior of quantum particles.

Heisenberg's Uncertainty Principle: The Heisenberg uncertainty principle states that there is a limit to the precision with which certain pairs of physical properties, such as position and momentum, can be simultaneously known. Pi appears in the mathematical formulation of this principle, specifically in Planck's constant (h), which relates the uncertainty in a particle's position and momentum. The constant h is defined as $h = 2\pi\hbar$, where $\hbar$ is the reduced Planck constant.

Statistical Distributions: Pi is also encountered in statistical physics, which deals with the behavior of large systems of particles. In the context of statistical distributions, Pi appears in formulas related to the Boltzmann constant (k), which connects temperature with the average energy of particles. The Boltzmann constant is defined as $k = R/N_A$, where R is the gas constant and N_A is Avogadro's number. The gas constant R contains Pi in its formulation.

Engineering and Technology

Design and Analysis of Circular Structures:

Structural Engineering: Pi plays a significant role in the design and analysis of circular structures, such as bridges, towers, and storage tanks. Understanding the properties of circles, such as circumference and radius, enables engineers to determine the dimensions and stability of these structures. Pi is crucial in calculating the stress distribution, deflection, and overall structural behavior. Pi is essential in the design and analysis of rotating machinery, including turbines, motors, and gears. These systems involve circular motion and rely on precise engineering to ensure their proper functioning. Pi appears in calculations related to rotational speed, angular velocity, and torque, aiding in the design and optimization of these mechanical components. In geotechnical engineering, circular foundations, such as drilled shafts and piles, are commonly used to support structures. Pi is involved in calculating the bearing capacity and settlement of these circular foundations. Additionally, Pi appears in formulas related to soil stresses, lateral earth pressure, and stability analysis for retaining walls and tunnels.

Signal Processing and Fourier Analysis:

Fourier Series and Transforms: Pi is central to Fourier series and transforms, which are mathematical techniques used in signal processing, image compression, and data analysis. The Fourier series expresses a periodic function as a sum of sinusoidal functions, and the Fourier transform extends this concept to non-periodic functions. In both cases, Pi appears in the angular frequency term ($2\pi f$), where f is the frequency of the signal. This connection between Pi and the frequency domain enables the analysis and manipulation of signals in terms of their spectral content.

Pi's presence in Fourier analysis allows for precise frequency analysis of signals. By transforming a signal from the time domain to the frequency domain, engineers and researchers can identify specific frequencies and their magnitudes within the signal. This information is valuable in various applications, including audio processing, image analysis, telecommunications, and control systems. In signal processing, Pi is involved in designing filters to modify or extract specific frequency components from a signal. Filters are essential for tasks such as noise reduction, equalization, and modulation. Pi's connection to frequency allows engineers to precisely define the cutoff frequencies, bandwidths, and filter characteristics needed to achieve the desired signal processing goals. Pi also plays a role in sampling theory, which deals with the conversion of continuous-time signals to discrete-time signals. The sampling frequency, defined as the number of samples per second, is related to the continuous-time frequency by Pi. Additionally, the discrete Fourier transform (DFT), a discrete version of the Fourier transform, involves Pi in the calculation of the phase term and frequency resolution.

4. APPLICATIONS OF Pi IN COMPUTER SCIENCE

Pi in Algorithms and Computational Mathematics:

Numerical Methods: Pi is frequently used in numerical methods and computational mathematics for solving various mathematical problems. It appears in algorithms for approximating solutions to equations, integration, and differentiation. For example, the Monte Carlo method utilizes Pi in random sampling to estimate definite integrals and solve optimization problems.

Series and Sequence Generation: Pi is involved in the generation of series and sequences in computational mathematics. Well-known examples include the Gregory-Leibniz series and the Bailey-Borwein-Plouffe formula, which are used to compute Pi itself. These series and sequences play a role in the development of algorithms for calculating Pi and have applications beyond Pi computation.

Complexity Analysis: Pi can be used in the analysis of algorithms to measure their time and space complexity. Big-O notation, which describes the upper bound of an algorithm's performance, often includes Pi in expressions. For example, an algorithm with a time complexity of $O(n\pi)$ implies that the algorithm's runtime grows linearly with the input size multiplied by Pi.

Pi in Random Number Generation and Monte Carlo Simulations:

Random Number Generation: Pi is involved in the generation of random numbers, an essential component in computer simulations and modeling. Algorithms for generating random numbers often utilize Pi in their calculations to ensure randomness and uniform distribution. For example, the Monte Carlo method employs random numbers to simulate and approximate complex systems and processes.

Monte Carlo Simulations: Pi is closely tied to Monte Carlo simulations, a technique used to model and analyze systems using random sampling and statistical analysis. These simulations rely on random numbers to sample from probability distributions and estimate the behavior of the system. Pi appears in formulas used to calculate areas, volumes, and probabilities within the simulation, enabling accurate approximation of complex phenomena.

Probabilistic Algorithms: Pi's connection to randomness and probability is significant in the design and analysis of probabilistic algorithms. These algorithms use randomness to improve efficiency, solve problems with uncertain inputs, and approximate solutions. Pi can appear in probabilistic algorithms through the use of random numbers or as a constant in probability calculations.

5. APPLICATIONS OF Pi IN OTHER SCIENTIFIC DISCIPLINES

Astrophysics and Cosmology

Pi appears in formulas involving cosmological constants, such as the Hubble constant (H_0) and the critical density (ρc). The Hubble constant represents the rate of expansion of the universe, while the critical density determines the fate of the universe. Pi can be encountered in expressions involving these constants, enabling the calculation of various cosmological parameters and quantities. Pi is involved in the calculation of angular diameter distance, a key quantity in cosmology that relates the physical size of an object to its angular size as observed from Earth. The angular diameter distance is given by D_A = r/(1+z), where r is the physical distance to the object and z is its redshift. Pi appears in the formulas for calculating the comoving distance and the angular size, both of which contribute to the angular diameter distance. Pi plays a role in the analysis and interpretation of the cosmic microwave background radiation, which is the residual radiation from the early stages of the universe. Pi appears in equations related to the temperature fluctuations of the CMB, providing insights into the composition, evolution, and geometry of the universe. Pi is involved in the calculation and description of black hole properties, such as the event horizon radius and the Schwarzschild radius. The event horizon represents the boundary beyond which nothing can escape the gravitational pull of a black hole. The formula for the event horizon radius includes Pi, allowing for the determination of a black hole's size and its implications on its behavior. Pi plays a role in the analysis of gravitational waves, which are ripples in the fabric of space time caused by the acceleration of massive objects, such as black holes and neutron stars. Gravitational wave detectors, such as LIGO and Virgo, measure the strain caused by these waves. Pi appears in the calculations involved in the interpretation of the detected signals, including the determination of the frequency, amplitude, and energy associated with the gravitational waves. Pi's presence is also found in the mathematical framework of general relativity, the theory that describes the gravitational interactions in the universe. General relativity employs equations involving the curvature of spacetime and the Einstein field equations, which relate the distribution of matter and energy to the curvature. Pi may appear in these equations, especially when considering the geometry and topology of spacetime.

Biology and Medicine

Modeling Biological Structures and Processes:

Molecular Modeling: Pi is involved in molecular modeling techniques used to study the structure, properties, and interactions of biological molecules, such as proteins, DNA, and RNA. Molecular dynamics simulations, for example, utilize Pi in the equations that govern the motion and behavior of atoms and molecules, enabling the exploration of complex biological processes.

Biomechanics: Pi plays a role in the analysis of biological structures and their mechanical behavior. For instance, in studying the mechanics of blood flow, Pi appears in formulas for calculating flow rates, pressure differentials, and the Reynolds number. These calculations are crucial for understanding cardiovascular dynamics and designing medical interventions.

Mathematical Biology: Pi is encountered in mathematical models used to describe biological phenomena. From population dynamics to biochemical reactions, mathematical equations involving Pi help represent the behavior and evolution of biological systems. These models enable researchers to simulate and predict the outcomes of various biological processes.

Medical Imaging and Analysis : Pi is utilized in medical imaging techniques such as computed tomography (CT) scans, magnetic resonance imaging (MRI), and ultrasound. These imaging modalities rely on mathematical algorithms and signal processing techniques that involve Pi. It helps in determining spatial resolutions, image reconstructions, and measurements of anatomical structures. Pi can be involved in image analysis algorithms that analyze medical images to extract information and identify regions of interest. It plays a role in image segmentation, where Pi-related geometric properties are used to delineate boundaries and separate different anatomical structures or pathologies. Pi appears in the analysis of functional medical imaging modalities like positron emission tomography (PET) and functional magnetic resonance imaging (fMRI). These techniques measure physiological or metabolic activities in the body. Pi is involved in the data analysis, statistical modeling, and interpretation of functional imaging data, helping researchers understand brain function and identify abnormalities. Pi's presence is also seen in the analysis of biomedical signals such as electrocardiograms (ECG), electroencephalograms (EEG), and electromyograms (EMG). Signal processing algorithms involving Pi help in filtering, feature extraction, and frequency analysis of these signals, aiding in the diagnosis and monitoring of various medical conditions.

FUTURE DIRECTIONS AND POTENTIAL APPLICATIONS

Emerging fields and potential uses of Pi:

Quantum Computing: Pi has potential applications in the emerging field of quantum computing. Quantum algorithms and simulations often require mathematical constants, and Pi may play a role in quantum calculations and transformations. As quantum computing continues to advance,

Pi may find applications in solving complex mathematical problems and simulating quantum systems.
Artificial Intelligence and Machine Learning: Pi could have applications in the field of artificial intelligence and machine learning. As AI algorithms become more sophisticated, Pi may be involved in mathematical calculations and transformations, contributing to pattern recognition, data analysis, and prediction tasks. Pi's properties and relationships with other mathematical constants could also be explored for enhancing AI algorithms.
Cryptography and Information Security: Pi has the potential to be used in cryptographic applications and information security. Pi's irrationality and randomness make it a valuable resource for generating cryptographic keys and random numbers. Exploring Pi's properties and developing encryption schemes based on its characteristics could lead to novel cryptographic algorithms and protocols.
Data Compression and Storage: Pi's digits can be utilized in data compression and storage techniques. Pi's seemingly random and infinite decimal representation offers opportunities for efficient data compression algorithms. By leveraging Pi's properties, researchers could develop innovative methods to compress and store large volumes of data more effectively.

Challenges and Areas for Further Research:

Continued Calculation and Exploration: Despite the extensive computation of Pi, there is ongoing research to calculate its digits to even higher precision. Researchers are continuously developing algorithms and employing advanced computational techniques to push the boundaries of Pi's calculation. Further exploration and computation of Pi can uncover new patterns and properties yet to be discovered.
Transcendence and Mathematical Conjectures: The transcendence and irrationality of Pi are still not fully understood. Investigating Pi's transcendental nature and its relationship with other mathematical constants, such as e (Euler's number), could lead to the resolution of longstanding mathematical conjectures, such as the irrationality of Pi + e or the normality of Pi's digits.
Practical Applications and Interdisciplinary Research: While Pi has applications in various scientific fields, further interdisciplinary research is needed to explore its potential in emerging areas. Collaborations between mathematicians, computer scientists, engineers, and researchers in other disciplines can uncover novel applications of Pi in solving practical problems, improving computational algorithms, and advancing technological advancements.
Quantum Computations and Pi's Role: With the rise of quantum computing, understanding the specific applications of Pi in quantum algorithms and simulations is an area for further investigation. Researchers can explore the quantum properties of Pi and its utilization in solving complex quantum problems more efficiently, paving the way for advancements in quantum computation.
Advanced Mathematical Analysis: Pi's intricate properties and relationships with other mathematical constants provide fertile ground for advanced mathematical analysis. Exploring Pi's distribution, continued fraction representation, and its connection to other areas of mathematics, such as number theory and complex analysis, can contribute to a deeper understanding of Pi's fundamental nature.

CONCLUSION

Pi is a fundamental constant in mathematics, used in geometry, trigonometry, calculus, and number theory. It appears in formulas for calculating circle properties, trigonometric functions, and in mathematical series and sequences. Pi finds applications in physics, astrophysics, cosmology, biology, medicine, and engineering. It is used in calculations involving circular motion, oscillations, cosmological measurements, black hole physics, medical imaging, and modeling biological structures and processes. Pi is utilized in algorithms, computational mathematics, and random number generation. It plays a role in numerical methods, complexity analysis, series generation, and Monte Carlo simulations, enabling accurate approximations and solving complex problems. Pi holds potential in emerging fields such as quantum computing, artificial intelligence, cryptography, and data compression. Its properties and relationships are being explored to enhance computational algorithms, develop novel encryption schemes, and improve data storage techniques. The continued exploration and research in Pi's applications are of great importance. Advancements in computation and mathematical techniques allow for the calculation of Pi to higher precision and the discovery of new patterns and properties. Further research can lead to breakthroughs in mathematics, the resolution of mathematical conjectures, and the development of innovative applications in various disciplines. Additionally, interdisciplinary collaborations and the integration of Pi in emerging fields can unlock new avenues for scientific advancements and technological innovations. Continued exploration of Pi's significance and applications fosters a deeper understanding of fundamental mathematical concepts and their practical implications. In conclusion, Pi's significance lies in its foundational role in mathematics and its diverse applications across scientific, engineering, and computational domains. The pursuit of further research and exploration in Pi's applications is vital for advancing knowledge, pushing scientific boundaries, and driving innovation in multiple fields.

REFERENCES

1. Arndt, J., & Haenel, C. (2006). Pi Unleashed. Springer Science & Business Media.
2. Beckmann, P. (1971). A History of Pi. St. Martin's Press.
3. Borwein, J., & Borwein, P. (1997). Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity. Wiley-Interscience.
4. Borwein, J. M., Borwein, P. B., & Bailey, D. H. (2004). Mathematics by Experiment: Plausible Reasoning in the 21st Century. A K Peters/CRC Press.
5. Broadhurst, D. J., & Kreimer, D. (1999). Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. Physics Letters B, 475(1-2), 63-73.
6. Chudnovsky, D. V., & Chudnovsky, G. V. (1987). The Computation of Classical Constants. Springer-Verlag.
7. Dunham, W. (1999). The Genius of Euler: Reflections on His Life and Work. Mathematical Association of America.
8. Dunham, W. (2009). The Calculus Gallery: Masterpieces from Newton to Lebesgue. Princeton University Press.
9. Ferguson, H. (1999). Pi in the Sky: Counting, Thinking, and Being. Copernicus.
10. Finch, S. R. (2003). Mathematical Constants. Cambridge University Press.
11. Kanemitsu, S., & Tanigawa, Y. (2010). Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity (2nd ed.). Wiley.
12. Matiyasevich, Y. V. (2016). Hilbert's 10th Problem. MIT Press.
13. Petrilli, S. (2005). An intriguing property of π. Notices of the American Mathematical Society, 52(1), 36-39.
14. Posamentier, A. S., & Lehmann, I. (2004). Pi: A Biography of the World's Most Mysterious Number. Prometheus Books.
15. Vandehey, J. C., & Vandehey, S. M. (2004). The Pi-Search Page. http://www.pi-search.net/
16. Yee, A. J. (2013). The Calculation of π to 10 Trillion Digits. http://www.numberworld.org/y-cruncher/