INTERNATIONAL ADVANCE JOURNAL OF ENGINEERING, SCIENCE AND MANAGEMENT (IAJESM)
July-December 2023, Submitted in December 2023, iajesm2014@gmail.com, ISSN -2393-8048
Multidisciplinary Indexed/Peer Reviewed Journal. SJIF Impact Factor 2023 =6.753

Study on Inclusion Relation Among Λ-Statistical Convergence and Strong Almost (V,Λ)-Summability with Strong Almost Convergence

KAPIL, Former Students, Department of Mathematics, Kurukshetra University, Kurukshetra, Haryana (India)
Email: kapilkaswan409@gmail.com

Abstract:

In this research we aim to unify some known results. In the process we pull together much of what is known about this topic and we will simplify some of their existing proofs. As a consequence we provide an unified point of view which allows us to solve several unsolved questions. In fact, we will obtain results in the caontext of ideal convergence. We will show that under reasonable conditions on a given non-trivial ideal, the studied properties do not depend on the ideal that we use to define the convergence spaces associated to the wuc series. This allows us to extend our results for an arbitrary summability method that shares some kind of ideal-convergence on the realm of all bounded sequences. This will allow us to unify the known results and obtain answers to some unresolved questions. The research is organized as follows.

Keywords: Λ-Statistical Convergence, Strong Almost (V,Λ)-Summability And Strong Almost Convergence

Introduction

Statistical Convergence,was published almost fifty years ago, has flatter the domain of recent research. Unlike mathematicians studied characteristics of statistical convergence and applied this notion in numerous extent such as measure theory, trigonometric series, approximation theory, locally compact spaces, and Banach spaces,etc. The present thesis emphasis on certain results studied by Ferenc Mo´ricz in his two research researchs i.e., "Statistical Convergence of Sequences and Series of Complex Numbers with applications in Fourier Analysis and summability " and in "Statistical Limit of Lebesgue Measurable functions with ∞ with applications in Fourier Analysis and summability". The perception of conjunction has been generalized in various ways through different methods such as summability and also a method in which one moves from a sequence to functions. In 1932 earlies, Banach coined the first generalization of it and named as "almost convergence". Later it was studied by Lorentz in 1948 [1].

The most recent generalization of the classical convergence i.e.,a new type of conjunction named as Statistical Convergence had been originated first via Henry Fast[3] in 1951. He characterizes this hypothesis to Hugo Steinhaus[19]. Actually, it was Antoni Zygmund[20] who evince the results, prepositions and assertion on Statistical Convergence in a Monograph in 1935. Antoni Zygmund in 1935 demonstrated in his book "Trigonometric Series" where instead of Statistical convergence he proposes the term "almost convergence" which was later proved by Steinhaus and Fast([19] and [3]).

Then, Henry Fast[3] in 1951 developed the notion analogous to Statistical Convergence, Lacunary Statistical Convergence and λ Statistical Convergence and it was reintroduced by Schoenberg[18] in 1959.Since then the several research research related to the concept have been published explaining the notion of convergence and is applications.The objective of the study is to discuss the fundamentals and results along with various extensions which have been subsequently formulated [2].

A sequence (xn) in a Banach space X is said to be statistically convergent to a vector L if for any $\varepsilon > 0$ the subset $\{n : kxn - Lk > \varepsilon\}$ has density 0. Statistical convergence is a summability method introduced by Zygmund [1] in the context of Fourier series convergence. Since then, a theory has been developed with deep and beautiful results [2] by different authors, and moreover at the present time this theory does not present any symptoms of abatement. The theory has important applications in several branches of Applied Mathematics (see the recent monograph by Mursaleen [3]). It is well known that there are

ROOTINDEXING JOURNAL ABSTRACTING AND INDEXING SERVICE
INTERNATIONAL ADVANCE JOURNAL OF ENGINEERING, SCIENCE AND MANAGEMENT (IAJESM)
July-December 2023, Submitted in December 2023, iajesm2014@gmail.com, ISSN -2393-8048
Multidisciplinary Indexed/Peer Reviewed Journal. SJIF Impact Factor 2023 =6.753

results that characterize properties of Banach spaces through convergence types. For instance, Kolk [4] was one of the pioneering contributors. Connor, Ganichev and Kadets [5] obtained important results that relate the statistical convergence to classical properties of Banach spaces.

In this section we study some inclusion relations between almost λ-statistical convergence, strong almost (V,λ)-summability and strong almost convergence. First we show that every strongly almost summable sequence is almost statistically convergent.

Theorem 1.4.1. If a sequence $x = \{\xi_k\}$ is almost strongly summable to ξ, then it is almost statistically convergent to ξ.

Proof. Suppose that $x = \{\xi_k\}$ is almost strongly summable to ξ. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in m.} \qquad \ldots(1)$$

Let us take some ε > 0. We have

$$\sum_{k=1}^{n} |\xi_{k+m} - \xi| \geq \sum_{|\xi_{k+m}-\xi| \geq \varepsilon} |\xi_{k+m} - \xi|$$

$$\geq \varepsilon|\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| \geq \varepsilon \lim_{n\to\infty} \frac{1}{n}|\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Hence by (1) and the fact that ε is fixed number, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in m.}$$

$\Rightarrow$ x is almost statistically convergent.

Theorem 1.2.2. Let $\lambda = \{\lambda_n\}$ be same as defined earlier. Then

(i) $\xi_k \to \xi[\hat{V},\lambda] \Rightarrow \xi_k \to \xi(\hat{S}_\lambda)$

and the inclusion $[\hat{V},\lambda] \subseteq \hat{S}_\lambda$ is proper,

(ii) if $x \in l_\infty$ and $\xi_k \to \xi(\hat{S}_\lambda)$, then $\xi_k \to \xi[\hat{V},\lambda]$ and hence $\xi_k \to \xi[\hat{c}]$ provided $x = \{\xi_k\}$ is not eventually constant.

(iii) $\hat{S}_\lambda \cap l_\infty = [\hat{V},\lambda] \cap l_\infty$,

where l_∞ denotes the set of bounded sequences.

Proof. (i). Since $\xi_k \to \xi[\hat{V},\lambda]$, for each ε > 0, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in m.} \qquad \ldots(2)$$

Let us take some ε > 0. We have

$$\sum_{k\in I_n} |\xi_{k+m} - \xi| \geq \sum_{\substack{k\in I_n \\ |\xi_{k+m}-\xi| \geq \varepsilon}} |\xi_{k+m} - \xi| \geq \varepsilon|\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| \geq \varepsilon \lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Hence by using (2) and the fact that ε is fixed number, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in m,}$$

i.e. $\xi_k \to \xi(\hat{S}_\lambda)$.

It is easy to see that $[\hat{V},\lambda] \subsetneq \hat{S}_\lambda$.

(ii). Suppose that $\xi_k \to \xi(\hat{S}_\lambda)$ and $x \in l_\infty$. Then for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \qquad \text{uniformly in m.} \qquad \ldots(3)$$

Since $x \in l_\infty$, there exists a positive real number M such that $|\xi_{k+m} - \xi| \leq M$ for all k and m. For given $\varepsilon > 0$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| = \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |\xi_{k+m}-\xi| \geq \varepsilon}} |\xi_{k+m} - \xi| + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |\xi_{k+m}-\xi| < \varepsilon}} |\xi_{k+m} - \xi|$$

$$\leq \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |\xi_{k+m}-\xi| \geq \varepsilon}} M + \frac{1}{\lambda_n}\sum_{k\in I_n} \varepsilon$$

$$= \frac{M}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{\lambda_n}[n-(n-\lambda_n+1)+1]$$

$$= \frac{M}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{\lambda_n}\lambda_n$$

$$= \frac{M}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| \leq M \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon$$

Hence by using (3), we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| = 0 \qquad \text{uniformly in m.} \qquad \ldots(4)$$

$$\Rightarrow \qquad \xi_k \to \xi[\hat{V}, \lambda].$$

Further, we have

$$\frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| = \frac{1}{n}\sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{n}\sum_{k=n-\lambda_n+1}^{n} |\xi_{k+m} - \xi|$$

$$= \frac{1}{n}\sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{n}\sum_{k\in I_n} |\xi_{k+m} - \xi|$$

$$\leq \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi|$$

$$\leq \frac{2}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi|$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| \leq 2 \lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi|$$

Hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in m.} \qquad \text{[Using (4)]}$$

$$\Rightarrow \qquad \xi_k \to \xi[\hat{c}].$$

(iii). Let $x \in l_\infty$ be such that $\xi_k \to \xi(\hat{S}_\lambda)$.
Then by (ii),

$\xi_k \to \xi[\hat{V},\lambda]$.

Thus

$$\hat{S}_\lambda \cap l_\infty \subset [\hat{V},\lambda] \cap l_\infty. \quad \ldots(5)$$

Also by (i), we have

$$\xi_k \to \xi[\hat{V},\lambda] \Rightarrow \xi_k \to \xi(\hat{S}_\lambda).$$

So $\quad [\hat{V},\lambda] \subset \hat{S}_\lambda$.

$$\Rightarrow \quad [\hat{V},\lambda] \cap l_\infty \subset \hat{S}_\lambda \cap l_\infty. \quad \ldots(6)$$

Hence by (5) and (6)

$$\hat{S}_\lambda \cap l_\infty = [\hat{V},\lambda] \cap l_\infty.$$

This completes the proof of the theorem

References:

1. P. Schaefer, Infinite matrices and invariant means, *Proc. Amer. Math. Soc.* 36 (1) (1972) 104-110.
2. I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* 66 (1959) 361-375.
3. H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* 2 (1951) 73-74.
4. Zygmund, *Trigonometric Series*, Second Ed., Cambridge Univ. Press, Cambridge, UK, 1979.
5. M. A. Akcoglu and L. Sucheston, Weak convergence of positive contractions implies strong convergence of averages, Probability Theory and Related Fields, 32 (1975) 139 – 145.
6. M.A. Akcoglu and A. del Junco, Convergence of averages of point transformations, Proc. Amer. Math. Soc. 49 (1975) 265-266. [7] I. Assani, Pointwise convergence of averages along cubes, preprint 2003, Math. DS/0305388, arxiv.org.

I. Assani, Pointwise convergence of nonconventional averages, Colloq. Math. 102 (2005) 245- 262.

7. P. Billingsley, Ergodic Theory and Information, John Wiley & Sons, New York, 1965.
8. T.J. I'A Bromwich and T.M. MacRobert, An Introduction to the Theory of Infinite Series, 3rd ed. Chelsea, New York, 1991.