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Existence of Positive Solution for The Eighth Order Boundary Value Problem Using Classical Version of Leray-Schauder Alternative Fixed-Point Theorem

Syed Muzaffer Hussain Dehbendhi, Dept. of Mathematics, Vishwabharti Women's Degree College, Rainawari, Srinagar (Kashmir)

ABSTRACT

Fixed point theorems are developed for single-valued and set-valued mappings of metrics paces, topological vector spaces, posets and lattices, Banach lattices etc., Among the themes of fixed point theory, the topic of approximation of fixed points of mappings is particularly important because it is useful for proving the existence of fixed points of mappings. It can be applied to prove the solvability of differential equations, optimization problems, variational inequalities, and equilibrium problems.

Due to the importance of fixed point theory and the high volume of active research in the nonlinear analysis, we have done investigations in this field.

The most physical phenomena can be converted into mathematical problems. For simplicity, one can rewrite the mathematical problem into fixed point problem/iterative equation. To find the solution to the original problem, it is enough to find the fixed point for new (iterative) equation.

INTRODUCTION

B.Ahmad and S.K.Ntouyas [AN12] conferred some existence results based on some standard fixed point theorems and Leray-Schauder degree theory for an nth-order nonlinear differential equation with four-point nonlocal integral boundary conditions. Motivated by these studies, in this chapter we have investigated the existence of solutions of eighth order boundary value problem

$$\begin{cases} y^{(8)}(x)=\Phi(x,y(x),y''(x)), & 0<x<1; \\ y(0)=y'(0)=y''(0)=y'''(0)=y^{(4)}(1)=y^{(5)}(1)=y^{(6)}(1)=y^{(7)}(1)=0, \end{cases} \quad (1)$$

Where $\Phi \in C([0,1]\times R\times R,R)$ and R=(1,1).

We considered the following eighth order boundary value problem under the assumption that $\Phi\in C([0,1]\times R\times R,R)$. E=C([0,1]) with the norm

$\|y\|=\max\{|y|_\infty, |y''|_\infty\}$ where $|y|_\infty=\max_{0\leq x\leq 1}|y(x)|$ for any $y\in E$.

Q.Yao [Yao10] obtained Green's function for fourth order differential equations. In this chapter, we derived Green's function for eighth order differential equations. The following Lemma is obtained by using the concept of Lemma 1

Lemma 1. *Let* $f\in C[0,1]$. *Then the following eighth order boundary value problem*

$$\begin{cases} y^{(8)}(x)=f(x), & 0<x<1 \\ y(0)=y'(0)=y''(0)=y'''(0)=y^{(4)}(1)=y^{(5)}(1)=y^{(6)}(1)=y^{(7)}(1)=0, \end{cases} \quad (2)$$

Has the integral formulation

$$y(x)=\int_0^1 G(x,s)f(s)ds$$

where $G:[0,1]\times[0,1]\to[0,1)$ *is the Green's function. By chapter 2, the Green's function for eighth order boundary value problem is given by*

$$G(x,s)=\frac{1}{5040}\begin{cases} s^4[(x-s)^3+4x(x-s)^2+10x^2(3x-s)], \\ x^4[(s-x)^3+4s(s-x)^2+10s^2(3s-x)], & 0\leq x<s<1, \end{cases} \quad (3)$$

Lemma 2. *For all* $(x,s)\in[0,1]\times[0,1]$, *we have* $0\leq G(x,s)\leq G(s,s)$.

Proof. The proof is obvious, so we leave it.

Define the integral operator T:E→Eby

$$T(y(x))=\frac{1}{5040}\int_0^1 s^4[(x-s)^3+4x(x-s)^2+10x^2(3x-s)]f(s)ds$$
$$+\frac{1}{5040}\int_0^1 x^4[(s-x)^3+4s(s-x)^2+10s^2(3s-x)]f(s)ds$$

By Lemma1, the boundary value problem (1) has a solution if the operator

T has a fixed point in E. Hence to find the solution of given boundary value problem, it is enough to find the fixed point for the operator T in E. Since T is compact and hence T is completely continuous.

Let (E,||.||) *be a Banach space*, U ⊂ E *be an open Bounded subset such that* 0∈U *and* T:U→E*be a completely continuous operator. Then*

(1)either T *has a fixed point in* U, *or*

(2)there exist an element x ∈∂u *and a real number* λ >1 *such that* x=T(x).

Main Results

Here, we proved some important results which will help to prove the existence of a nontrivial solution for the eighth order boundary value problem (1). Consider Φ∈ C([0,1]×R×R,R)

Theorem 1.*Suppose that* Φ(x,0,0)=0 *and there exists nonnegative functions* p, q,r ∈L^1[0,1] *such that*

|(x,y,z)|p(x)|y|+q(x)|z|+r(x), a.e. (x,y,z)∈[0,1]×R×R,

and

$$[5s_7+s_6+5s_4][p(s)+q(s)]ds<1.$$

Then the boundary value problem (1) *has at least one nontrivial solution* y∈E.

Proof. Let

$$A=\frac{1}{720}\int_0^1[5s^7+s^6+5s^4][p(s)+q(s)]ds,\ 0$$

$$B=\frac{1}{720}\int_0^1[5s^7+s^6+5s^4]r(s)ds.$$

By hypothesis, we have A<1. Since(x,0,0)=0, there exists an interval [a,b]⊂[0,1] such that $\min_{a\le x\le b}$|(x,0,0)|>0 and as (x)≥ Φ|(x,0,0)| a.e. x∈[0,1].

Hence B>0:

Let L=B(1-A)$^{-1}$ and U={y∈E:||y||<L}. Assume that y∈∂u and λ >1 are such that Ty=λy. Then

λL=λ||y||=||Ty||

$$=\max_{0\le x\le 1}|(Ty)(x)|$$

$$\le\frac{1}{5040}\int_0^x s^4\ [(x-s)^3+4x(x-s)^2+10x^2(3x-s)]|\Phi(s,y(s),y''(s))|ds$$
$$+\frac{1}{5040}\int_x^1 x^4[(s-x)^3+4s(s-x)^2+10s^2(3s-x)]|\Phi(s,y(s),y''(s))|ds$$
$$\le\frac{1}{5040}\max_{0\le x\le 1}\int_0^x s^4[(x-s)^3+4x(x-s)^2+10x^2(3x-s)]|\Phi(s,y(s),y''(s))|ds$$
$$+\frac{1}{5040}\max_{0\le x\le 1}\int_x^1 x^4[(s-x)^3+4s(s-x)^2+10s^2(3s-x)]|\Phi(s,y(s),y''(s))|ds$$
$$`=\frac{1}{5040}\int_0^1 s^4[(1-s)^3+4(1-s)^2+10(3-s)]|\Phi(s,y(s),y''(s))|ds$$

$+\frac{1}{5040}\int_0^1 s^4[s^3+4s(s)^2+10s^2(3-s)]|\Phi(s,y(s),y''(s))|ds$

$=\frac{1}{5040}\int_0^1[34s^7+7s^6-21s^5+35s^4]|\Phi(s,y(s),y''(s))|ds$

$\leq\frac{1}{5040}\int_0^1[35s^7+7s^6+35s^4]|\Phi(s,y(s),y''(s))|ds$

$\leq\frac{1}{720}\int_0^1[5s^7+s^6+5s^4][p(s)|y(s)|+q(s)|y''(s)|+r(s)]ds$

$\leq\frac{1}{720}\int_0^1[5s^7+s^6+5s^4][p(s)\max_{0\leq s\leq 1}|y(s)|+q(s)\max_{0\leq s\leq 1}|y''(s)|+r(s)]ds$

$\leq\frac{1}{720}\int_0^1[5s^7+s^6+5s^4][p(s)|y|_1+q(s)|y''|_1+r(s)]ds$

$\leq\frac{1}{720}\int_0^1[5s^7+s^6+5s^4][p(s)kyk+q(s)\|y\|+r(s)]ds$

$=\frac{1}{720}\int_0^1[[5s^7+s^6+5s^4][p(s)+q(s)]]\|y\|+\frac{1}{720}\int_0^1 [5s^7+s^6+5s^4]r(s)ds$

$=A\|y\|+B=AL+B.$

Hence, $\lambda L\leq AL+B$

$$\lambda\leq A+\frac{B}{L}=A+\frac{B}{B(1-A)^{-1}}=A+(1-A)=1,$$

which is a contradiction, since $\lambda>1$, hence by Theorem3.2.1, T has a fixed point y $\in$U. Since $\Phi(x,0,0)=0$, the boundary value problem(3.1) has a nontrivial solution y $\in$E.

Theorem 2. *Let* $\Phi(x,0,0)=0$ *and there exists non-negative functions* p, q, r $\in L^1[0,1]$ *such that*

$|\Phi(x,y,z)|p(x)|y|+q(x)|z|+r(x)$ i.e. $(x,y,z)\in[0,1]\times R\times R$.

Assume that one of the conditions given below is satisfied

(1)There exists a constant k>-5 *such that*

$P(s)+q(s)\leq\frac{720(8+k)(7+k)(5+k)}{11k^2+148k+495}s^k,\ a.e.\ 0\leq s\leq 1,$

$$\mu\left\{s\in[0,1]:P(s)+q(s)<\frac{720(8+k)(7+k)(5+k)}{11k^2+148k+495}s^k\right\}>0$$

where μ *is a measure.*

(2)There exists a constant k>-1 *such that*

$P(s)+q(s)\leq\frac{6\prod_{i=1}^8(k+i)}{k^3+21k^2+152k+594}(1-s)^k,\ a.e. 0\leq s\leq 1,$

$$\mu\left\{s\in[0,1]:p(s)+q(s)<\frac{6\prod_{i=1}^8(k+i)}{k^3+21k^2+152k+594}(1-s)^k\right\}>0$$

Where is μ *a measure.*

(3)There exists a constant a>1 *such that*

$$\int_0^1[p(s)+q(s)]^a<\left[\frac{1}{\frac{1}{144}\left(\frac{1}{7b+1}\right)^{\frac{1}{b}}+\frac{1}{720}\left(\frac{1}{6b+1}\right)^{\frac{1}{b}}+\frac{1}{144}\left(\frac{1}{4b+1}\right)^{\frac{1}{b}}}\right]^a,\quad\left(\frac{1}{a}+\frac{1}{b}=1\right)$$

Then the boundary value problem (1) *has at least one nontrivial solution* y*$\in$E.

Proof. To prove this theorem it is enough to prove A<1.

Let

(1)Consider, $A=\frac{1}{720}\int_0^1[5s7+s6+5s^4][p(s)+q(s)]ds$

$A=\frac{1}{720}\int_0^1[5s7+s2+5s4][p(s)+q(s)]ds<\frac{720(8+k)(7+k)(5+k)}{11k^2+148k+495}\left[\frac{1}{720}\int_0^1 (5s^7+s^6+5s^4)s^k ds\right]$ $=\frac{720(8+k)(7+k)(5+k)}{11k^2+148k+495}\left[\frac{1}{720}\int_0^1(5s^{7+k}+s^{6+k}+5s^{4+k})ds\right]$

$$=\frac{720(8+k)(7+k)(5+k)}{11k^2+148k+495}\left[\frac{1}{720}\left(\frac{5}{8+k}+\frac{1}{7+k}+\frac{5}{5+k}\right)\right]$$

$$=\frac{720(8+k)(7+k)(5+k)}{11k^2+148k+495}\left[\frac{11k^2+148k+495}{720(8+k)(7+k)(5+k)}\right.$$

Thus, A<1.

(2) in this case, we have

$$A=\frac{1}{720}\int_0^1[5s^7+s^6+5s^4][p(s)+q(s)]\,ds<\frac{6\prod_{i=1}^{8}(k+i)}{k^3+21k^2+152k+594}\left[\frac{1}{720}\int_0^1(5s^7+s^6+5s^4)(1-s)^k ds\right]$$

$$<\frac{6\prod_{i=1}^{8}(k+i)}{k^3+21k^2 152k+594}\left[\frac{1}{720}\int_0^1 5s^7(1-s)^k ds+\int_0^1 s^6(1-s)^k ds+\int_0^1(5s^4(1-s)^k ds\right.$$

$$<\frac{6\prod_{i=1}^{8}(k+i)}{k^3+21k^2+152k+594}\left[\frac{1}{720}\int_0^1 5s^2(1-s)^k ds+\int_0^1 s6(1-s)^k ds+\int_0^1(5s^4(1-s)^k ds\right]$$

$$<\frac{6\prod_{i=1}^{8}(k+i)}{[k^3+21k^2+152k+594]}\left[\frac{120}{\prod_{i=1}^{5}(k+i)}+\frac{720}{\prod_{i=1}^{7}(k+i)}+\frac{720\times 35}{\prod_{i=1}^{8}(k+i)}\right]$$

$$=\frac{6\prod_{i=1}^{8}(k+i)}{k^3+21k^2+152k+594}\left[\frac{k^3+21k^2+152k+594}{6\prod_{i=1}^{8}(k+i)}\right]=1$$

Therefore, A <1.

(3)By Hölder inequality, we have

$$A\le\left[\int_0^1(p(s)+q(s))^a\,ds\right]^{\frac{1}{a}}.\left[\frac{1}{144}\left(\int_0^1(s^7)^b\,ds\right)^{\frac{1}{b}}+\frac{1}{720}\left(\int_0^1(s^6)^b\right)^{\frac{1}{b}}+\frac{1}{144}\left(\int_0^1(s^4)^b ds\right)^{\frac{1}{b}}\right]$$

$$A\le\left[\int_0^1(p(s)+q(s))^a\right]^{\frac{1}{a}}.\left[\frac{1}{144}\left(\frac{1}{7b+1}\right)^{\frac{1}{b}}+\frac{1}{720}\left(\frac{1}{6b+1}\right)^{\frac{1}{b}}+\frac{1}{144}\left(\frac{1}{4b+1}\right)^{\frac{1}{b}}\right]$$

$$<\left(\frac{1}{\frac{1}{144}\left(\frac{1}{7b+1}\right)^{\frac{1}{b}}+\frac{1}{720}\left(\frac{1}{6b+1}\right)^{\frac{1}{b}}+\frac{1}{144}\left(\frac{1}{4b+1}\right)^{\frac{1}{b}}}\right)$$

$$\left[\frac{1}{144}\left(\frac{1}{7b+1}\right)^{\frac{1}{b}}+\frac{1}{720}\left(\frac{1}{6b+1}\right)^{\frac{1}{b}}+\frac{1}{144}\left(\frac{1}{4b+1}\right)^{\frac{1}{b}}\right]$$

= 1.

DISCUSSION

Here we have given some examples to verify the above results.

Example1. Consider,

$$\begin{cases} \mathbf{Y^{(8)}(x)}=\frac{x^5}{2}y\sin\sqrt{y}+\frac{\sqrt{x}}{3}y''\cos y''-5+e^{2x}, \quad \mathbf{0\le x\le 1,} \\ y(0)=y'(0)=y''(0)=y'''(0)=0, \\ Y^{(4)}(1)=y^{(5)}(1)=y^{(6)}(1)=y^{(7)}(1)=0. \\ \text{Set} \end{cases}$$

$\Phi(x,y,z)=\frac{x^5}{2}y\sin\sqrt{y}+\frac{\sqrt{x}}{3}z\cos z-5+e^{2x}$,using classical version of Leray-Schauder alternative fixed point theorem

$P(x)=\frac{x^5}{2}$, $q(x)=\frac{\sqrt{x}}{3}$, $r(x)=5+e^{2x}$.

One can easily verify that p,q,r∈$L^1[0,1]$ are nonnegative functions, and

$|\Phi(x,y,z)|=|\frac{x^5}{2}y\sin\sqrt{y}+\frac{\sqrt{x}}{3}z\cos z-5+e^{2x}|$

$\leq$p(x)|y|+q(x)|z|+r(x),

a.e.(x,y,z)∈[0,1]×R×R.

Also,

A=$\frac{1}{720}\int_0^1[5s^7+s^6+5s^4][p(s)+q(s)]ds$

$=\frac{1}{720}\int_0^1[5s^7+s^6+5s^4]\left(\frac{s^5}{2}+\frac{s^{\frac{1}{2}}}{3}\right)ds$

$=\frac{1}{720}\int_0^1[\frac{5}{2}s^{12}+\frac{5}{3}s^{\frac{15}{2}}+\frac{s^{11}}{2}+\frac{s^{\frac{13}{2}}}{3}+\frac{5}{2}s^9+\frac{5}{3}s^{\frac{9}{2}}]ds$

$=\frac{899251}{630115200}<1.$

Thus, by Theorem 1, the boundary value problem (1) has at least one non-trivial solution y*∈E.

Example 2. Consider the problem,

$$\begin{cases} Y^{(8)}(x)\frac{y^4}{(5+4y^3)\sqrt{x}}\cos y+\frac{4(y'')^3}{7\sqrt{x}}+\frac{2y''}{\sqrt{x}}-\cos\sqrt{x}, & 0\leq x\leq 1,\\ Y(0)=y'(0)=y''(0)=y'''(0)=0, \\ y^{(4)}(1)=y^{(5)}(1)=y^{(6)}(1)=y^{(7)}(1)=0. \end{cases}$$

Set

$\Phi(x,y,z)=\frac{y^4}{(5+4y^3)\sqrt{x}}\cos\cos y+\frac{4z^3}{7\sqrt{x}}+\frac{2x}{\sqrt{x}}-\cos\sqrt{x},$

$P(x)=\frac{1}{5\sqrt{x}}$, $q(x)=\frac{4}{7\sqrt{x}}+\frac{2}{\sqrt{x}}$, $r(x)=\cos\sqrt{x}$.

One can easily verify that p,q,r∈L^1[0,1] are nonnegative functions, and

$$|\Phi(x,y,z)|=\left|\frac{y^4}{(5+4y^3)\sqrt{x}}\cos y+\frac{4z^3}{7\sqrt{x}}+\frac{2z}{\sqrt{x}}-\cos\sqrt{x}\right|$$

$\leq$ p(x)|y|+q(x)|z|+r(x), a.e. (x,y,z)Є[0,1] × R×R.

Hence (2) reduces to

Let k=$-\frac{1}{2}$ >-5. Then,

$$\frac{720(8+k)(7+k)(5+k)}{11k^2+148k+495}=\frac{631800}{1695}$$

Therefore,

$$p(s)+q(s)=\frac{1}{5\sqrt{s}}+\frac{4}{7\sqrt{s}}+\frac{2}{\sqrt{s}}=\frac{97}{35}s^{-\frac{1}{2}}<\frac{621800}{1695}s^{-\frac{1}{2}}$$

$$\mu\left\{s\in[0,1]:p(s)+q(s)<\frac{720(8+k)(7+k)(5+k)}{11k^2+148k+495}s^k\right\}>0$$

Where μ is a measure. Thus by the Theorem:2 assumption (1), the boundary value problem (2) has at least one nontrivial solution y*⊂E.

Example 3. Consider the problem,

$$\begin{cases} y^{(8)}(x)=\frac{y^3}{4(3+y^4)\sqrt[3]{(1-x)^2}}\sin y+\frac{(y'')^2}{(5+y'')\sqrt[3]{(1-x)^2}}e^{2x}+\sin 3x, 0\leq x\leq 1,\\ Y(0)=y'(0)=y''(0)=0,\\ Y^{(4)}(1)=y^{(5)}(1)=y^{(6)}(1)=y^{(7)}(1)=0. \end{cases}$$

Set

$\Phi(x,y,z)=\frac{y^3}{4(3+y^4)\sqrt[3]{(1-x)^2}}\sin y+\frac{(y'')^2}{(5+y'')\sqrt[3]{(1-x)^2}}e^{2x}+\sin 3x$

$P(x)=\frac{1}{4\sqrt[3]{(1-x)^2}}, q(x)=\frac{1}{5\sqrt[3]{(1-x)^2}}, r(x)=e^{2x}+\sin 3x.$

Here we can easily prove that p,q,r⊂L^1[0,1] are nonnegative functions, and

$|\Phi(x,y,z)|=\left|\frac{y^3}{4(3+y^4)\sqrt[3]{(1-x)^2}}\sin y+\frac{z^2}{(5+z)\sqrt[3]{(1-x)^2}}+e^{2x}+\sin 3x\right|$

$\leq$ p(x)|y|+ q(x)|z| + r(x), a.e. (x, y, z)Є[0,1] × R × R.

Let a=4 >b $=\frac{4}{3}$ >1.We have that $\frac{1}{a}+\frac{1}{b}$=1. Then

$$\int_0^1 (p(s)+q(s))^{a} ds = \int_0^1 [\,4\sqrt[4]{2+s}\,]^4 ds = 640.$$

Also, we have

$$\left[\frac{1}{\frac{1}{144}\left(\frac{1}{7b+1}\right)^{\frac{1}{b}}+\frac{1}{720}\left(\frac{1}{6b+1}\right)^{\frac{1}{b}}+\frac{1}{144}\left(\frac{1}{4b+1}\right)^{\frac{1}{b}}}\right]^{a} = \left[\frac{1}{\frac{1}{144}\left(\frac{3}{31}\right)^{\frac{3}{4}}+\frac{1}{720}\left(\frac{1}{9}\right)^{\frac{3}{4}}+\frac{1}{144}\left(\frac{3}{19}\right)^{\frac{3}{4}}}\right]^4$$

≈9406732117.352

Therefore

$$\int_0^1 (p(s)+q(s))^{a} < 9406732117.3529$$

Further, by Theorem 2 assumption (*3*), *the boundary value problem* (3:4:4) *has at least one nontrivial solution* y*ЄE.

CONCLUSION

In this paper, we obtained the results to prove the existence of positive solution for the eighth order boundary value problem with the help of classical version of Leray-Schauder alternative fixed point theorem. By applying these results, one can easily verify that whether the given boundary value problem is solvable or not.

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