A Study of Optimality Condition and Duality have Assumed A Critical Part in the Advance of Mathematical Programming.

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Abstract: It Optimization theory is a standout amongst the hugest and captivating branches of connected mathematics. It is formally worried about the procedure of maximization or minimization of a coveted capacity while fulfilling the overall constraints. This has caught practically whole domain of human advance. Truth be told, nature has a galore of circumstances where optimum system status are produced. In metals and alloys, the atoms take places of minimum vitality to form unit cells. These unit cells characterize crystalline structure of materials. Genetic mutation for survival is another case of nature's optimization procedure. Like nature, human organizations have additionally worked hard towards discovering perfection. Arrangements of their problems have been looked for the most part on the premise of involvement and judgment. Be that as it may, in the world of today, the expanded rivalry and buyer demands regularly require optimum arrangements instead of simply doable arrangements. It has been encountered that optimization of design process spares cash for a organization by essentially decreasing the improvement time. In this manner the theory of optimization manages picking the best option among a few choices in the feeling of given capacity with minimum conceivable assets. This creates a class of problems named as mathematical programming problems. The optimum looking for techniques are known as mathematical programming techniques and by and large concentrated as a piece of operations research.

Keywords: Mathematical programming, Karush-Kuhn-Tucker optimality, Notations

1.1 Introduction

In this study, Optimization theory is a standout amongst the hugest and captivating branches of connected mathematics. It is formally worried about the procedure of maximization or minimization of a coveted capacity while fulfilling the overall constraints. This has caught practically whole domain of human advance. Truth be told, nature has a galore of circumstances where optimum system status are produced. In metals and alloys, the atoms take places of minimum vitality to form unit cells. These unit cells characterize crystalline structure of materials. Genetic mutation for survival is another case of nature's optimization procedure. Like nature, human organizations have additionally worked hard towards discovering perfection. Arrangements of their problems have been looked for the most part on the premise of involvement and judgment. Be that as it may, in the world of today, the expanded rivalry and buyer demands regularly require optimum arrangements instead of simply doable arrangements.

Mathematical programming possessed a status of logical field in its own particular right amid late 1940's and from that point forward it has experienced gigantic advancement. It is presently considered as a standout amongst the most energetic and energizing branches of modem mathematics having broad applications in different settings, for example, designing, financial matters and common sciences. An exceptionally regular case of a mathematical programming problem shows up in discovering minimum weight design of structure subject to constraints on stress and deflection.

The form of a mathematical programming problem is as follows,

(MP): Optimize (minimize/maximize)f(x).

Subject to

$g_i(x) \le 0, i=1,2,3,\ldots,m,$

$h_j(x)=0, j=1,2,3,\ldots,k,$

$x \in X$

Here the function f and each fj and hj are genuine esteemed capacities characterized on n dimensional Euclidean space R^n and $X \subseteq R^n$. This is alluded to as the general mathematical programming problem. The constraints, $gi(x) \leq 0$, i = 1, 2…, m are alluded to as to as inequality constraints, the constraints $hj(x) = 0$, j = 1, 2... k are called fairness constraints. The incorporation $x \in X$ is known as a conceptual constraints. On the off chance that the goal and imperative capacities are differentiable then we portray the above problem as differentiable program. On the off chance that the goal and the inequality constraints are relative capacity and X is a convex set, at that point the above problem is known as a convex programming problem.

1.2 PRELIMINARIES

1.2.1 Notations

R^n = n-dimensional Euclidean space,

R^n = The non-negative orthant in R^n,

M^T = Transpose of the matrix M,

Let θ be a numerical function defined on an open set ɽ in R^n, then

$\nabla \theta (X)$ denotes the gradient of θ at $\bar{x}$, that is

$$\nabla\theta(\bar{x}) = \left(\frac{\partial\theta(\bar{x})}{\partial x^1},\dots,\frac{\partial\theta(\bar{x})}{\partial x^n}\right)^T$$

Let ψ a real valued twice continuously differentiate function defined on an open set contained in $R^n x R^m$. Then $\nabla_x \psi$ (x, y) and $\nabla_y \psi$ f(x, y) denote the gradient (column) vector of f with to x and y respectively,

$$\nabla_x\Psi(\bar{x},\bar{y}) = \left(\frac{\partial\psi}{\partial x^1},\frac{\partial\psi}{\partial x^2},\dots,\frac{\partial\psi}{\partial x^n}\right)^T_{(\bar{x},\bar{y})}$$

$$\nabla_y\Psi(\bar{x},\bar{y}) = \left(\frac{\partial\psi}{\partial x^1},\frac{\partial\psi}{\partial x^2},\dots,\frac{\partial\psi}{\partial x^n}\right)^T_{(\bar{x},\bar{y})}$$

Further $\nabla^2_{xx}(\bar{x}\bar{y}\ )$ and $\nabla^2_{xy}\psi(x,y)$ denote respectively the (n x n) and (n x m) matrices of second order partial derivative i.e

$$\nabla^2_{xx}\ \psi(\bar{x},\bar{y}) = \left(\frac{\partial^2\psi}{\partial x^i \partial x^j}\right)_{(\bar{x},\bar{y})},$$

$$\nabla^2_{xy}\Psi(\bar{x},\bar{y}) = \left(\frac{\partial^2\psi}{\partial x^i, \partial y^j}\right)_{(\bar{x},\bar{y})}.$$

The symbols and $\nabla^2_{yy}\psi$ $(\bar{x}\bar{y}$)are characterized correspondingly. Be that as it may, at certain places, to make the importance of the setting all the more clear, the subscripts of∇and ∇^2 are brought as the variable as for which the function is being separated.

1.2.2 Definitions

Definition 1.1 Let $X \subseteq c\ R^n$ be an open and convex set and $f : X \rightarrow R$ be differentiable. Then we define f to be.

1. **Convex**, if for all $x_1, x_2 \in X$,

 $f(x_1) - f(x_2) \geq (x_1 - x_2)^T \nabla f(X_2)$.

2. Strictly convex, if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$

 $f(x_1)-f(x_2) > (x_1-x_2)^T \nabla f(x_2)$.

3. Quasiconvex, if for all $x_1, x_2 \in X$,

 $f(x_1) \leq f(x_2) \Rightarrow (x_1 - x_2)^T \nabla f(X_2) \leq 0$.

4. Pseudoconvex, if for all $x_1, x_2 \in X$,

$(x_1 - x_2)^T \nabla f(x_2) > 0 \Rightarrow f(x_1) > f(x_2)$.

5. Strictly Pseudoconvex, if for all $x_1, x_2 \in X$ $x_1 \neq x_2$

$(x_1 - x_2)\nabla(x_2) \geq 0 \Longrightarrow\rightarrow f(x_1) > f(x_2)$.

6. Invex, if there exists a vector function $\eta : R^n \times R^n \rightarrow R^n$

Such that for all $x_1, x_2 \in X$, $f(x_1) - f(x_2) \geq \eta(x_1, x_2)^T \nabla f(x_2)$.

7. Pseudoinvex, if there exist a vector function $\eta : R^n \times R^n \rightarrow R^n$

such that for all$x_1 x_2 \in X$,$(x_1, x_2)^T \nabla f(x_2) \geq 0 \Longrightarrow f(x_1) \geq f(x_2)$

8. Quasiinvex, if there exist a vector function $\eta : R^n \times R^n \rightarrow R^n$

such that for all $x_1\ x_2 \in X$, $f(x_1) \leq f(x_2) \Longrightarrow\rightarrow \eta(x_1, x_2)^T \nabla f(x_2) \leq 0$.

9.Second order convex (Bonvex), if for al $x_1, x_2 \in R^n$. $f(x_1) - (x_2) \geq (x_1 - x_2)^T \nabla^2 f(x_2)p - \frac{1}{2}(p^T \nabla^2 f(x^2)p)$.

10. Second order pseudoconvex (Pseudobonbex), if for all $x_1, x_2 \in X$, $p \in R^n$

$(x_1 - x_2)^T \nabla^2 f(x_2) + (x_1 - x_2)^T \nabla^2 f(x^2)p \geq 0 \Longrightarrow\rightarrow f(x_1)\ f(x_2) - (p^T \nabla^2 f(x_2)p)$.

Unmistakably, a differentiable convex, pseudoconvex, quasiconvex function is invex, pseudoinvex or quasiinvex separately with $\eta(x_1, x_2) = (x_1 - x_2)$. Promote we characterize f to be concave. Strictly concave, quasconcave, pseudoconcave, strictly pseudocovex, on X according as - f is convex, strictly convex, quasi convex, pseudo convex, strictly pseudo convex.

Definition 1.2; Let $f : R^n \rightarrow R$ be a convex function, then a sub-gradient of f at point $X \in R^n$ is a vector $\xi \in R^n$ satisfying.

$$f(y) \geq f(x) + \xi^T(y - x), \forall y \in R^n.$$

Definition 1.3: The set of all sub gradients of f at $x \in R^n$ is called sub-differential of f at X is denoted by $\partial f(x)$.

Definition 1.4: Let Γbe a nonempty of R^n.

(i) The set Γ is called a cone if
$x \in, \Gamma\ \lambda \geq 0 \Rightarrow \lambda x \in .\Gamma$

(ii) A cone $\Gamma \subseteq R^n$ is convex if
$x + y \in , \Gamma$ for all $x,y \in . \Gamma$

(iii) Let $\Gamma \subseteq c\ R^n$ be a convex cone. Then $\Gamma *$ defined as
$\Gamma * = \{ z \in R^n : z^T X \leq 0$, for all $x \in \Gamma$ is called the polar cone of Γ.

Definition 1.5: Let $X \subseteq R^n$ and $Y \subseteq R^m$ be convex subsets and g: $X \times Y \rightarrow R$. Then the function g is said to be convex - concave on X x Y if it is convex in 'x' for each fixedy $\in$ Y and concave in y for each fixed $x \in X$

There are number of constraint qualifications, which are required to be fulfilled by the constraints, while building up the fundamental optimality criteria to guarantee that specific Lagrange multipliers exist and are non-zero. Here we portray just four of them for fulfillment of ideas.

(i)Slater's Constraint Qualification: Let X° is a convex set in R^n. The m-dimensional convex vector function g on X° which characterizes the convex achievable region

$X = \{x : x \in X^{\circ}\ g(x) \leq 0\}$ is said to fulfill Slater's constraint qualification on X^0 if there exists a $\bar{x} \in X$ with the end goal that $g(\bar{x}) < 0$.

(ii) **The Kuhn-Tucker's Constraint Qualification**: Let X° be an open set in R^n, Let g be a m-dimensional vector function defined on X°and let $X = \{x: x \in x_0\ g(x) \leq 0\}$. At that point the constraints are said to fulfill the Kuhn-Tucker's Constraint. Qualification at $\bar{x} \in X$ if g is differentiable at $\bar{x}$ and if

$y \in R^n$, $\nabla g_i(\bar{x})y \le 0$ $\Rightarrow$ There exists an n-dimensional vector function e defined in the interval [0,1] such that

(a) $e(0) = \bar{x}$

(b) $e(\tau) \in X$ for $0 \le \tau \le 1$.

(c) e is differentiable at $\tau = 0$ and $\frac{de(0)}{d\tau} = \lambda y$ for some $\lambda > 0$.

where $I = \{i : g_i(\bar{x}) = 0\}$.

(iii) The reverse convex constraint qualification: Let X^n be an open set in R^n, let g be m-dimensional vector function defined on x_0, and let $X = \{x | x \in X°, g(x) \le 0\}$, g is said to fulfill the reverse convex constraint qualification at $\bar{x} \in X$, if g is differentiable at $\bar{x}$, and if for each is $i \in I$ either g, is concave at $\bar{x}$ or gi is linear on R^n, where $I = \{i | g_i(\bar{x}) = 0\}$.

Linear independence constraint qualification: The condition that the vectors $\nabla g_i(x_0)$........,$\nabla g_m(x_0)$ is linearly independent is often alluded to as linear independence constraint qualification.

REVIEW OF RELATED WORK

1.3.1 Duality in Differentiable Mathematical Programming

Let $f: R^n \to R$ and $h_j : R^n \to R$, $(j = 1,2,\ldots m)$then consider the nonlinear programming issue:

(P): Min f(x)

subject to,

$h_j(x) \le 0$,$(j = 1,2,\ldots,m)$.

For $\lambda \in R^m$ the Lagrangian dual for issue (P) is defined as

$$\text{(LD):}\quad \underset{\lambda \ge 0}{\text{Max}}\left(\underset{x \in R^n}{\text{Min}}\left(f(x) + \lambda^T h(x)\right)\right)$$

That is,

(LD): M inf(u) + $\lambda^T h(u)$

Subject to,

$$f(u) + \lambda^T h(u) = \underset{x \in R^n}{\text{Min}}\ f(x) + \lambda^T h(x),\ \ \lambda \in R^m_+$$

In the event that all the function f and hj : (j = 1,2,. ..,m) are the differentiable convex functions, at that point the issue (LD) is comparable to the following issue:

(WD): M ax $f(x) + \lambda^T h(x)$

Subject to

$V(f(x) + \lambda^T h(x)) = 0, \lambda \ge 0, \lambda \in R^m$.

This is nothing yet the Wolfe sort dual for the issue (P). Mangasaria explained by implies of a case that certain duality theorems may not be legitimate if the goal or the constraint function is a summed up convex function. This spurred Mond and Weir to introduce an alternate dual for (P) as

(MWD): Maxf(x)

Subject to
$\nabla(f(x) + \lambda^T h(x)) = 0$,
$\lambda^T h(x) \geq 0$,
$\lambda \geq 0, \lambda \in R^m$,
Also, they demonstrated different duality hypotheses under pseudoconvexity off and quasiconvexity $\lambda^T h$ of for all doable arrangement of (P) and (M-WD). Later Weir and Mond inferred adequacy of Fritz John optimality criteria under pseudoconvexity of the goal and quasiconvexity or semi-strict convexity of constraint functions. They planned the accompanying double utilizing the Fritz John optimality conditions rather of the Karush-Kuhn - Tucker optimality conditions and demonstrated different duality hypotheses in this way the necessity of constraint qualification is wiped out.
(F,D): Maximize f(x)
subject to
$\lambda_0 \nabla f(x) + \nabla \lambda^T h(x) = 0$
$\lambda^T h(x) \geq 0$
$(\lambda_0,\lambda) \geq 0$, $(\lambda_0,\lambda) \neq 0$.

1.5 Conclusions

The results, obtained in this thesis are presented in chapters 2-7, are briefly summarized as follows:
First section part is separated in to two segments. In Section 2.1 we consider the following non differentiable nonlinear problem with help functions:
(NP) : Minimize f (x) + s (x| C)
Subject to
gj(x) + s (x | D j)≤ 0, j = l , 2 .....,m.
w h e r e
(i) for the n-dimensional Euclidean space R^n, f; $R^n \rightarrow R$ and g_1 ,: $R^n \rightarrow R$, (j = 1, 2 ,…, m), are continuously differentiable, and
(ii) s (.| C) and s (.| Dj), (j = 1, 2,…, m) are respectively the support functions of
convex compact sets C and Dj, (j = 1, 2, ..., m) in R^n.
For this problem, we present the following mixed type dual (Mix D) to (N P):

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