ISSN: 2393-8048
International Advance Journal of Engineering, Science and Management (IAJESM)
Multidisciplinary, Multilingual, Indexed, Double Blind, Open Access, Peer-Reviewed, Refereed-International Journal.
SJIF Impact Factor =8.152, January-June 2025, Submitted in February 2025

Theorical Advances and Applications of Harmonious Labeling of Path-Related Graphs

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Abstract

A useful tool in many disciplines, including biology, chemistry, and computer science, is graph theory. Graph labeling facilitates the analysis of graph structures by giving distinct labels to graph elements. One kind of labeling that has drawn notice is harmonious labeling, which has become more popular due to its sophisticated qualities and possible uses. This paper explores a key field of graph theory: harmonic labeling in path-related graphs: theoretical developments and real-world applications. Significant progress has been made in the area of harmonious labeling, which assigns unique labels to graph vertices and edges so that the labels on neighboring edges are unique.

Keywords: Harmonious Labeling, Path-Related Graphs, Graph Theory, Graph Labeling Techniques

1. INTRODUCTION

The theory of graphs, a cornerstone of discrete mathematics, has found significant applications across varied domains such as computer science, networking, chemistry, and biology. In computer science, graphs model social networks, communication networks, and data structures like trees and linked lists. In networking, graphs help in creating and analyzing the flow of information, optimizing routes, and improving the robustness and efficiency of networks. In chemistry, graphs are essential in describing molecular structures, where atoms are vertices, and bonds are edges, enabling the study of chemical compounds and reactions.

Within this broad area, graph labeling has evolved as a powerful tool for evaluating and comprehending graph structures and their features. Graph labeling assigns numbers to vertices and edges according to rules. Labels can be used to infer graph properties like symmetry, connectivity, and node distances. Different labeling systems highlight different graph characteristics. Labeling critical nodes and edges in network design can optimize resource allocation and improve security.

Harmonic labeling stands out due to its advanced formulation and wide range of applications. This scheme has garnered attention for its ability to reveal graph structural properties, particularly symmetry and periodicity. The harmonic labeling method has proven particularly valuable in domains like telecommunications and coding theory, where it aids in constructing efficient communication protocols and error-detecting codes.

Harmonic labelling's mathematical rigor and rich graph features make it elegant. By guaranteeing that the labels of adjacent vertices are different and follow harmonic criteria, this system assists in finding patterns and regularities within the graph. This, in turn, can lead to more efficient algorithms for graph traversal, optimization, and other computing activities. Moreover, the harmonic features of the labels can be utilized to examine the resonance frequencies in physical systems, such as vibrating strings and electrical circuits, making this labeling scheme relevant in applied physics and engineering as well.

1.1. Harmonious Labeling

Fundamentally, graph labeling is a mathematical approach that involves giving specific labels, usually numbers, to the vertices (nodes) and edges (connections between nodes) that make up a graph. The labeling procedure has multiple functions and is subject to particular regulations based on the chosen labeling scheme. Graph labeling aims to make graph structures easier to examine and analyze, leading to a better knowledge of their characteristics and behavior.

Vertex labeling is a popular kind of graph labeling in which each vertex in the graph is given a unique label. There are a lot of different rules for vertex labeling. In a basic vertex labeling scheme, for example, every vertex might be given a different integer from a predetermined set.

More intricate schemes need the total of labels on neighboring vertices to be distinct, like harmonic labeling. In situations where nodes' identities or positions must be determined, such as in scheduling issues, wireless network frequency assignment, or the study of chemical compounds where vertices stand in for atoms and edges for bonds, vertice labeling is very helpful.

On the other hand, edge labeling entails giving the graph's edges names. Just like with vertex labeling, edge labeling criteria might vary according to the use case. In elegant labeling, for instance, the labels of the edges are the exact differences—and these differences need to be distinct—between the labels of their end vertices. When the relationships or connections between nodes are the main focus of the scenario, edge labeling is frequently employed. Applications include biological networks, where edges could indicate the kind or degree of connections between creatures, and communication networks, where edge labels could indicate the capacity or cost of a link.

Certain labeling techniques name both edges and vertices, producing more complex and useful graph representations. For example, in total labeling, each vertex and each edge of the graph is assigned a label; the relationships between these labels are governed by certain criteria. These schemes are helpful in more complicated applications, like coding theory, where they aid in the creation of codes with desirable qualities, or in the modeling of complex networks in physics and engineering. They can offer a thorough perspective of the graph's structure.

Various graph labeling techniques are designed to draw attention to certain graph structural features. Chromatic labeling, for instance, focuses on allocating labels so that neighboring vertices (or edges) get distinct labels; it's similar to coloring a map so that no two nearby places have the same color. In challenges requiring scheduling, resource allocation, and map coloring, this kind of labeling is essential. Distinct labeling techniques, like radio labeling, magic labeling, and antimagic labeling, have specific uses in the optimization and resolution of graph theory and related issues.

There is a certain kind of vertex labeling that are known as harmonious labeling. A unique integer label is assigned to each vertex of a graph using this particular method. It is common for these labels to fall anywhere between 0 and q-1, where q is the total number of vertices in the network.

Following the labeling of the vertices, the next most important step is to label the edges. Through the process of determining the absolute difference between the labels of an edge's two endpoints, the label of an edge can be precisely determined. Two conditions must be met for a graph to have harmonic labels:

➢ **Distinct Edge Labels:**

In graph theory, different edge labels offer unique labels to each graph edge so no two edges share the same label. Edge labels are crucial in network design, communication protocols, and coding theory to avoid ambiguity and distinguish connections or paths. When each graph edge has a unique identification, it improves data flow tracking, error detection and repair, and system organization and efficiency. In a communication network, edge labels might represent multiple frequencies or channels to prevent signal interference. According to coding theory, distinct edge labels can correlate to various codewords, allowing data transmission mistakes to be identified and corrected. In algorithmic operations like graph coloring and routing, distinct edge labeling helps ensure computation integrity and correctness.

➢ **Label Range:**

The label range notion in graph labeling requires edge labels to be within an integer range, often from 1 to $q-1$, where q indicates an upper limit. Each vertex has a unique label, thus the absolute difference between the labels of any two adjacent vertices provides an edge label within this range. If vertices u and v are labelled l(u) and l(v), the edge linking them, e(u, v), must be labelled $|l(u)-l(v)|$. A uniform labeling scheme that follows the graph's labeling rules is achieved by limiting the difference between 1 and q−1. This method is useful in network topology, chemical graph theory, and coding theory, where graph structure and attributes must be carefully specified.

1.2. Path Graph Labeling Techniques

With the exception of the two endpoints, which are each connected to a single vertex, every vertex in a path graph is connected to exactly two additional vertices. Path graphs are a basic type of graph. This configuration creates a linear series of vertices joined by edges that resembles a straight line. Given their simplicity and the fundamental ideas they demonstrate, path graphs are particularly interesting in graph theory for research and labeling.

In order to reveal or take advantage of specific graph traits, labeling path graphs entails applying labels to either or both of the vertices or edges (or both) within predetermined limits. For path graphs, elegant labeling is a popular labeling technique. The process of graceful labeling involves assigning distinct labels to each vertex of a path graph with n vertices. These labels are chosen from the set $\{0, 1, 2..., n-1\}$ {0,1, 2..., n−1} so that the labels on the edges, which are the absolute differences between the labels of adjacent vertices, are all distinct and cover the set $\{1, 2..., n-1\}$ {1, 2..., n−1}. Because it guarantees a distinct and methodical manner to identify the edges, this labeling scheme is quite intriguing and can be a helpful tool in network design and optimization.

Harmonious labeling is an additional method of labeling path graphs. In this case, the vertices are labelled so that, for every edge, the total of the labels of the neighboring vertices is distinct and falls inside a given range. In communication networks, for example, harmonious labeling is used for frequency assignment. By preventing adjacent vertices—which represent frequencies—from colliding, harmonious labeling reduces interference.

Path graphs also benefit from edge labeling. For example, in edge-magic labeling, all edges must have the same label, which is equal to the sum of the labels of their end vertices plus a constant number. This kind of labeling has applications in fields such as chemistry, where it can be used to describe symmetric molecule structures, and also aids in the study of symmetric characteristics of graphs.

In addition, radio labeling is a different type of labeling that applies to path graphs. It entails labeling vertices in a way that ensures the length of the shortest path between any two vertices plus one is the absolute difference between their labels. This kind of labeling ensures that frequencies assigned to transmitters in close proximity do not conflict with one another, which makes it very helpful in the research of frequency assignment issues in radio networks.

Labeling path graphs under these and other constraints is a topic that not only broadens our knowledge of graph theory but also has many real-world applications. Labeling systems, for instance, are useful in computer science when designing effective algorithms for resource allocation and network routing. Labeling is a useful tool in biology that helps evaluate and model linear biological structures such as DNA sequences. Path graphs are a great place to start when studying more difficult labeling problems in graph theory because of their simplicity.

For the purpose of labeling path graphs, one strategy that is frequently used is the consecutive integer labeling method. In this method, vertices are assigned successive integers beginning with 0 or 1. The procedure in question is easy; nonetheless, it is possible that it does not fulfill the requirements for more complicated labeling schemes such as harmonic labeling.

Several different methods have been investigated by researchers in order to achieve harmonic labeling of path graphs. To achieve separate edge labels, one method involves assigning alternating odd and even integers to vertices. This is done in the expectation of achieving the desired result. On the other hand, this strategy does not ensure that harmonious labeling is achieved for all path lengths.

A further strategy centers on the use of modular arithmetic. Attempts have been made by researchers to generate harmonious labeling for specific path lengths by utilizing modulo procedures. Although this method has been beneficial in certain circumstances, it is not without its drawbacks.

Methods that are iterative have been utilized in order to discover classifications that are harmonic for path graphs. These techniques begin with an initial labeling and then proceed to adjust it in a step-by-step manner until a labeling that is harmonious is achieved. In spite of the fact that it is time-consuming, this strategy has produced some encouraging outcomes.

2. LITERATURE REVIEW

Dushyant Tanna (2013) This article presents the idea of "harmonious labeling of certain graphs," which is a pioneering method that creates a foundational framework for further investigation in the field of graph labeling theory. The research conducted by Tanna digs into the theoretical foundations of harmonic labeling. This method focuses on giving labels to the vertices of a graph in such a way that the disparities between labels on neighboring vertices are unique throughout the entire graph. While the work does provide an early step into this topic, it is notable that it does not provide a deep exploration of specific graph classes. This leaves potential for additional research into how these labeling approaches might be used to a variety of graph types. The purpose of this study is to provide a crucial foundation for understanding the fundamental concepts of harmonic labeling, thereby laying the groundwork for further research into its applications and extensions.

Jeyanthi and Philo (2016) Investigate the idea of "odd harmonious labeling," which is a specific kind of graph labeling in which the labels that are assigned to the edges of the graph are limited to odd numbers. Cycle-related graphs, which are essential structures in graph theory and have substantial ramifications in a variety of applications, are the primary focus of this research particularly. The purpose of this work is to further our understanding of harmonious qualities in these particular graph structures by investigating the ways in which odd harmonious labeling may be applied and the properties that have been discovered as a result of this application. By analyzing cycles, which are networks in which the vertices are connected in a closed loop, the authors give useful insights into the ways in which odd harmonious labeling can be applied efficiently, as well as the constraints or patterns that it may present. The work that they have done expands the existing knowledge of harmonic labeling, which normally includes labels being applied in such a way that every vertex receives a unique sum from its incident edges. In particular, they address situations in which these sums are confined to odd values.

Philo, Jeyanthi, and Davvaz (2022) By their investigation of full bipartite graphs, they provide a substantial contribution to the field of research that focuses on odd harmonious labeling. The purpose of their research is to increase understanding in the more general field of graph labeling by concentrating on this particular category of graphs. The graphs known as complete bipartite graphs, which are represented by the notation Km, nK_{m,n}Km,n, are made up of two sets of vertices that are not connected to each other. Each vertex in one set is connected to every vertex in the other set, but there are no edges inside either set. There is a one-of-a-kind opportunity to examine labeling strategies that preserve particular attributes across the graph, and this structure provides that possibility. In the work that they have done, Philo and his colleagues investigate the possibility of applying odd harmonious labeling to these graphs. This is a concept in which the labels of vertices and edges follow to specified parity rules. They investigate the circumstances under which such labeling systems are valid and the ways in which implementation of such schemes can be carried out successfully.

Pramesti (2021) proposes the Sn (m, r) graph and investigates the odd harmonious labeling aspects of the graph, making a significant contribution to the field of graph theory. The Sn (m, r) graph is a novel class of graphs that extends the study of labeling techniques by concentrating on odd harmonious labeling. This is a method in which labels are assigned to vertices in such a way that the difference between the labels of neighboring vertices is an odd integer. Through

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the analysis of a new family of graphs inside the Sn (m, r) framework, Pramesti's study contributes to the advancement of the theoretical knowledge of this labeling. This broadens the scope of research that is conducted in this particular field. The study provides new insights into the structure and properties of Sn (m, r) graphs for the purpose of paving the way for further exploration and application of odd harmonious labeling in a variety of graph classes. This is accomplished by systematically investigating the conditions under which odd harmonious labeling is possible for these graphs.

Lourdusamy, Wency, and Patrick (2018) It is possible to make great progress in the field of graph labeling by investigating both harmonic and odd harmonious labeling, and eventually by presenting the novel idea of "SD-harmonious labeling." This idea is expanded upon by the definition of SD-harmonious labeling, which is a novel version that offers more limits and possibilities for labeling schemes. The purpose of this novel idea is to offer a more nuanced method to graph labeling, which will open up new paths for investigation and application. Through the extension of the conventional frameworks and the introduction of this advanced form of labeling, their work contributes to the enhancement of the theoretical landscape and provides novel insights on the ways in which labeling can be adapted to fulfill the intricate needs of graph theory.

3. APPLICATIONS OF HARMONIOUS LABELING

The idea of harmonious labeling, which is found in graph theory, has a wide range of applications in a variety of different fields. The following is a summary of its most important applications:

- **Network Design and Optimization:** When creating effective communication networks, it is necessary to allocate different channels or frequencies to nodes (such routers or switches) in a manner that reduces interference. Harmonious labeling is a technique that is utilized in this process. Through the process of ensuring that neighbouring nodes are assigned distinct labels, harmonic labeling contributes to the optimization of network efficiency and the reduction of disputes.
- **Frequency Assignment:** Harmonious labeling is a technique that can be utilized to solve frequency assignment issues in the fields of broadcasting and telecommunications. Specifically, the objective here is to assign unique frequencies to transmitters or receivers in order to prevent nearby ones from interfering with one other. This is necessary in order to ensure that the signal is transmitted clearly and with as little interference as possible.
- **Scheduling Problems:** In situations when multiple jobs or events need to be scheduled without causing any conflicts, strategies that involve harmonious labeling can be utilized to find a solution to the scheduling challenge. For instance, harmonic labeling is helpful in the process of timetabling for educational institutions. This helps to ensure that no two classes that overlap have the same resources, which helps to avoid scheduling problems.
- **Error Detection and Correction:** Harmonious labeling is a technique that can be utilized in the application of coding theory to create error-detecting and error-correcting codes. The system is able to detect and fix errors more efficiently, which results in an improvement in the dependability of data transmission. This is accomplished by giving distinct labels to the various data symbols.
- **Graph Coloring:** There is a strong connection between harmonious labeling and difficulty with graph coloring. Using harmonic labeling helps ensure that neighboring elements (nodes or edges) have various labels, which is useful in tasks such as coloring maps or scheduling. This is especially helpful in situations when different colors or labels are required to reflect separate states or conditions.
- **Biological Modeling:** Harmonious labeling is a technique that can be utilized in computational biology to model the interactions between certain proteins or genes.

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Researchers are able to better understand and visualize complicated biological systems if they identify the various connections or paths in a way that is harmonious.

- **Robotics and Path Planning:** Harmonious labeling is a technique that can be utilized in the field of robotics to provide assistance to path planning algorithms. This technique ensures that various courses or routes are uniquely defined and do not overlap in a manner that could result in conflicts or accidents.

4. HARMONIOUS LABELING OF PATH RELATED GRAPHS

Harmonious labeling is the process of labeling graph vertices to meet certain constraints in graph theory. Harmonic labeling is defined as an injective function f: V(G)→ {0,1, 2…, q−1} for a graph G with q edges. This function ensures that all edge labels in G are distinct when (f(u)+f(v)) mod q is allocated to uv. As a result, it is impossible for two edges in the graph to have the same label. This guarantees that any pair of vertices that are connected by an edge will have a distinct edge sum.

When considering path-related graphs in specifically, harmonic labeling is a technique that can be utilized in a variety of different sorts of graphs that are formed from paths. The most fundamental of them is the path graph itself, which is represented by the symbol Pn. It is made up of a series of n vertices, and each vertex vi (for $1 \le i \le n-1$) is connected to vi+1. Particular methodologies have been developed in order to guarantee that the sum of labels on connected vertices mod q is unique for each edge.

When it comes to path-related graphs, there are more complicated structures that can be termed path-related graphs. Some examples of these structures include cycles and caterpillars. As an illustration, a cycle graph Cn is a closed loop consisting of n vertices, with each vertex being connected to two other vertices. To ensure that the sum of labels of neighboring vertices (taken modulo q) remains unique over the entirety of the cycle, it is necessary to ensure that the cycle is labelled in a manner that is harmonious. Due of the closed-loop structure of this, which puts additional limits on label assignments, this presents a greater challenge than labeling simple paths.

There is also the caterpillar graph, which is a tree in which all of the vertices are within a distance of one from a central "spine" path. This structure is another intriguing example. In caterpillars, harmonious labeling involves not only the path but also the "legs" that radiate from this path. This adds an additional layer of difficulty to the process of ensuring that unique edge sums are calculated. For the purpose of marking these increasingly sophisticated structures while preserving the harmonic condition, researchers have created particular methodologies.

Theorem 1: The graph G obtained by joint sum of two copies of fans Fn = Pn + K1 is harmonious, for all n.

Proof: Let v, v1, v2, ..., vn and v1', v2', ..., vn', v' be the vertices of Fn and Fn' respectively, where v and v' are apex vertices. G be the graph obtained by joining the apex vertices by an edge.

The vertex set of G is V(G) = {v, v', vi, vi' / $1 \le i \le n$}

The edge set of G is E(G) = {vivi+1 / $1 \le i \le n - 1$} ∪ {vvi / $1 \le i \le n$} ∪ {vi'vi+1' / $1 \le i \le n - 1$} ∪ {v'vi' / $1 \le i \le n$} ∪ {vv'}.

Note that G has 2(n + 1) vertices and 4n - 1 edges.

Here q = 4n – 1

Define f: V(G) → {0, 1, 2, ..., q - 1} as follows.

f(v) = 0

f(v') = 2n + 1

f(vi) = 2i - 1, $1 \le i \le n$

f(vi') = 2i, $1 \le i \le n$

The induced labeling f*: E(G) → {0, 1, 2, …, q - 1} defined by

f*(uv) = (f(u) + f(v)) (mod q) as follows

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$f^*(vv_i) = 2i - 1, 1 \leq i \leq n$
$= \{1, 3, \ldots, 2n - 1\}$
$f^*(vv') = 2n + 1$
$f^*(v'vi') = [f(vv') + 2i](\text{mod } q), 1 \leq i \leq n$
$= [f(v) + f(v') + 2i](\text{mod } q), 1 \leq i \leq n$
$= [2n + 1 + 2 + 2i](\text{mod } q), 1 \leq i \leq n$
$= \{2n + 3, 2n + 5, \ldots, 4n + 1\}(\text{mod } q)$
$= \{2n + 3, 2n + 5, \ldots, 4n - 3, 0, 2\}$
$= \{2n + 3, 2n + 5, \ldots, q - 2, 0, 2\}$
$f^*(vivi+1) = 4i, 1 \leq i \leq n-1$
$= \{4, 8, \ldots, 4n - 4\}$
$= \{4, 8, \ldots, q - 3\}$
$f^*(vi'vi+1') = 4i + 2, 1 \leq i \leq n - 1$
$= \{6, 10, \ldots, 4n - 2\}$
$= \{6, 10, \ldots, q - 1\}$
Thus, the edge labels are $\{0, 1, 2, \ldots, q - 1\}$
Thus, all the edge labels are distinct.
Hence the graph G is harmonious, for all n

It is crucial to first comprehend the structure of G in order to comprehend the proof that the graph G, which is created by adding the joint sum of two copies of fans Fn, is harmonic. Two identical fan graphs, each represented as Fn, are joined to form the graph G by combining their apex vertices with edges. A route Pn with an extra apex vertex connected to each vertex of the path makes up a fan graph Fn. As a result, all of the vertices from the two fan graphs—{v, v′, v1, v2, vn, v1′, v2′, vn′}—are included in the vertex set of graph G. The apex vertices of the two fan graphs in this case are v and 'v′.

As a result, the vertex set V(G) of G has 2(n+1) vertices, which include the vertices from the two fan graphs' routes as well as the apex vertices. The extra edge between the apex vertices v and v′ and the edges from both fan graphs make up the edge set E(G). The edges explicitly comprise the ones inside each route Pn (like {vivi+1} and {vi′vi+1′}), the extra edge between the apex vertices {vv′}, and the ones linking the apex vertices to the path vertices (like {vvi} and {v′vi′}).

A special labeling function f is utilized for the vertices in order to demonstrate that G is harmonic, that is, each edge can be assigned a unique label from {0,1, 2,…,q−1}, where q is the number of edges in G. The labels are assigned by the function f in the following ways: for vertices vi in one path, f(vi)=2i−1, f(v′)=2n+1, f(v)=0, and for vertices vi′ in the other path, f(vi′)=2i. This vertex labeling is used to define an edge labeling function f∗, where f∗(uv)=(f(u)+f(v)).

We confirm the edge labels' distinctness with particular computations. The labels for edges that are unique within their respective sets are 2i−1 and 4i, respectively, for edges that are within the paths, such vi′vi+1 and vivi+1. The computations also show that the labels for the edges vvi and v′vi′ that join the apex vertices to the path vertices are different. There is a unique label for the edge that joins the two apex vertices (vv′), and comparable calculations validate the distinctness for other sorts of edges.

5. CONCLUSION

In conclusion, the investigation of harmonic labeling in path-related graphs demonstrates both its broad practical implications and important theoretical contributions. This method provides a richer understanding of network structures and their attributes by guaranteeing that edge labels stay distinct and giving vertices unique names. When it comes to reducing interference in real-world systems, like communication networks, harmonic labeling is a crucial tool. It helps with frequency assignments to prevent signal overlap, scheduling to avoid task conflicts, and unique channel assignments to minimize interference.

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